Physics Problems

Problem 1 — The Unstable Coin on BMO

For a dance party in the Tree Fort, BMO is rigged up as a record player. Onyx sets a coin on BMO's flat top at a distance $R = 20$ cm from the axis of rotation, and BMO starts spinning from rest with a constant angular acceleration $\alpha_0 = 0.5$ rad/s^2. The coefficient of static friction between the coin and BMO is $\mu = 0.3$.

Given parameters:

- Distance from the axis: $R = 20$ cm
- Angular acceleration: $\alpha_0 = 0.5$ rad/s^2
- Static friction coefficient: $\mu = 0.3$

Task

Determine the time t at which the coin starts to slip.

Solution

The coin experiences two perpendicular acceleration components: the tangential acceleration $a_t = \alpha_0 R$ (constant) and the centripetal acceleration $a_c = \omega^2 R = \alpha_0^2 t^2 R$, which grows with time. The total required acceleration has magnitude:

$$a = R\sqrt{\alpha_0^2 + \omega^4} = R\alpha_0\sqrt{1 + \alpha_0^2 t^4} \tag{1}$$

Friction must supply this acceleration. Slipping begins exactly when the required force reaches the maximum static friction force μmg:

$$\mu g = R\alpha_0\sqrt{1 + \alpha_0^2 t^4} \tag{2}$$

Squaring and solving for the time:

$$1 + \alpha_0^2 t^4 = \left(\frac{\mu g}{R\alpha_0}\right)^2 \qquad \Rightarrow \qquad t = \frac{1}{\sqrt{\alpha_0}}\left[\left(\frac{\mu g}{R\alpha_0}\right)^2 - 1\right]^{1/4}$$

With the given numbers, $\mu g/(R\alpha_0) = 0.3 \times 9.8/(0.2 \times 0.5) = 29.4$, so:

$$\boxed{t = \frac{1}{\sqrt{0.5}}\sqrt[4]{29.4^2 - 1} \approx 7.7 \text{ s}} \tag{3}$$

Problem 2 — Marceline's Ring

Onyx borrows a thin circular music box ring of radius $R = 10$ cm that carries a uniformly distributed charge $Q = 1$ μC (moonlight-gathered starlight). A tiny Candy Kingdom love-bead of charge $q = -1$ nC and mass $m = 0.1$ g is placed at rest exactly at the center of the ring and then displaced an infinitesimal distance along the axis of the ring.

Given parameters:

- Ring radius: $R = 10$ cm
- Ring charge: $Q = 1$ μC

- Bead charge: $q = -1$ nC
- Bead mass: $m = 0.1$ g

Task

Show that the bead oscillates and determine its angular frequency ω for small displacements.

Solution

By symmetry the center of the ring is an equilibrium point. The electric potential on the axis at distance x from the center is

$$V(x) = \frac{kQ}{\sqrt{R^2 + x^2}} \tag{4}$$

Since the bead carries a charge opposite to the ring's, the potential energy is attractive:

$$U(x) = qV(x) = -\frac{k|q|\,Q}{\sqrt{R^2 + x^2}} \tag{5}$$

Expanding for small x:

$$U(x) \approx -\frac{k|q|Q}{R}\left(1 - \frac{x^2}{2R^2}\right) \tag{6}$$

Therefore

$$U''(0) = \frac{k|q|Q}{R^3} \tag{7}$$

and the bead performs simple harmonic motion with

$$\omega = \sqrt{\frac{U''(0)}{m}} = \sqrt{\frac{k|q|Q}{mR^3}} \tag{8}$$

Substituting numbers, $k = 8.99 \times 10^9\ \mathrm{Nm^2/C^2}$:

$$\omega = \sqrt{\frac{8.99 \times 10^9 \times 10^{-9} \times 10^{-6}}{10^{-4} \times 10^{-3}}} = \sqrt{89.9} \approx 9.5\ \mathrm{rad/s}$$

$$\boxed{\omega = \sqrt{\frac{k|q|Q}{mR^3}} \approx 9.5\ \mathrm{rad/s}} \tag{9}$$

Problem 3 — The Candy Kingdom Shuttle

In the Candy Kingdom, Onyx builds a sled across two long horizontal frictionless conducting rails separated by $\ell = 0.4$ m. Their left ends are connected through a resistor of resistance $R = 2\ \Omega$. A light conducting rod of negligible resistance and mass $m = 0.1$ kg rests across the rails. The whole Land of Ooo is bathed in a uniform vertical magnetic field $B = 0.5$ T. A constant horizontal force $F = 1$ N is applied to the rod, pulling it along the rails.

Given parameters:

- Separation of the rails: $\ell = 0.4$ m
- Load resistance: $R = 2\ \Omega$
- Rod mass: $m = 0.1$ kg

- Magnetic field: $B = 0.5$ T
- Applied force: $F = 1$ N

Task

Determine the terminal velocity of the rod.

Solution

As the rod moves with velocity v, the motional EMF is $\mathcal{E} = B\ell v$, driving a current $I = B\ell v/R$ through the circuit. The magnetic force on the rod opposes the motion and equals

$$F_m = I\ell B = \frac{B^2\ell^2}{R}\,v \tag{10}$$

The rod accelerates until this drag force balances the applied force:

$$F = \frac{B^2\ell^2}{R}\,v_t \qquad \Rightarrow \qquad v_t = \frac{FR}{B^2\ell^2} \tag{11}$$

With the given numbers:

$$v_t = \frac{1 \times 2}{0.5^2 \times 0.4^2} = \frac{2}{0.04} = 50 \text{ m/s}$$

$$\boxed{v_t = \frac{FR}{B^2\ell^2} = 50 \text{ m/s}} \tag{12}$$

Problem 4 — The Floor Lamp of the Pool

Gunter does late-night laps in Princess Bubblegum's pool. Onyx drops a tiny lamp to the bottom of the deep end, at depth $H = 2.0$ m. The index of refraction of water is $n = 1.33$. Looking down from above, the surface of the pool appears to glow within a bright circular patch.

Given parameters:

- Depth of the lamp: $H = 2.0$ m
- Index of refraction of water: $n = 1.33$

Task

Determine the radius r of the bright circle visible on the surface of the water.

Solution

Light rays leaving the lamp reach the water–air interface at an angle of incidence θ measured from the vertical. A ray escapes into the air only when θ is less than the critical angle, where total internal reflection begins:

$$\sin\theta_c = \frac{1}{n} \tag{13}$$

All rays with $\theta > \theta_c$ are totally reflected back into the pool and never reach an observer above. Hence the visible patch on the surface is a circle of radius r obtained from the geometry of the cone of escaping rays:

$$\tan\theta_c = \frac{r}{H} \tag{14}$$

Using $\tan\theta_c = \sin\theta_c/\sqrt{1-\sin^2\theta_c} = 1/\sqrt{n^2-1}$:

$$r = \frac{H}{\sqrt{n^2-1}} \tag{15}$$

Substituting the numbers:

$$r = \frac{2.0}{\sqrt{1.33^2-1}} = \frac{2.0}{\sqrt{0.7689}} \approx 2.28 \text{ m}$$

$$\boxed{r = \frac{H}{\sqrt{n^2-1}} \approx 2.3 \text{ m}} \tag{16}$$

Problem 5 — The Breathing Drum

In Flame Princess's insulated forge, Onyx seals a monatomic ideal gas ($\gamma = 5/3$) inside an upright vertical drum of cross-section $A = 0.01 \text{ m}^2$. A frictionless piston of mass $M = 5$ kg closes the drum from above; the ambient pressure is $P_0 = 10^5$ Pa. In equilibrium the piston sits at a height $h = 0.4$ m above the bottom of the drum.

Given parameters:

- Cross-section of the drum: $A = 0.01 \text{ m}^2$
- Piston mass: $M = 5$ kg
- Ambient pressure: $P_0 = 10^5$ Pa
- Equilibrium height: $h = 0.4$ m
- Monatomic gas: $\gamma = 5/3$

Task

The piston is pressed down a little and released. Determine the angular frequency ω of the small oscillations that follow.

Solution

If the piston is displaced by a small distance x from equilibrium (so that the gas volume changes by $\Delta V = Ax$), the adiabatic process obeys $PV^\gamma = \text{const}$, which gives

$$\frac{dP}{P} = -\gamma\frac{dV}{V} \qquad \Rightarrow \qquad dP = -\gamma P\frac{Ax}{Ah} = -\frac{\gamma P}{h}\,x \tag{17}$$

The net force on the piston, restoring it toward equilibrium, is

$$F = A\,dP = -\frac{\gamma P A}{h}\,x \tag{18}$$

This is a Hooke's-law restoring force, so the piston executes SHM with

$$\omega = \sqrt{\frac{\gamma P A}{Mh}} \tag{19}$$

The pressure at equilibrium balances the piston weight and the atmosphere:

$$P = P_0 + \frac{Mg}{A} \tag{20}$$

So the frequency becomes

$$\omega = \sqrt{\frac{\gamma(P_0 A + Mg)}{Mh}} \tag{21}$$

With the numbers ($P_0 A = 1000$ N, $Mg = 49$ N):

$$\omega = \sqrt{\frac{(5/3) \times 1049}{5 \times 0.4}} = \sqrt{874.2} \approx 30 \text{ rad/s}$$

$$\boxed{\omega = \sqrt{\frac{\gamma(P_0 A + Mg)}{Mh}} \approx 30 \text{ rad/s}} \tag{22}$$

Problem 6 — The Town Crier's Rope

A long, perfectly flexible uniform messenger rope of length $L = 20$ m and mass per unit length μ hangs vertically from the balcony of the Tree Fort, its lower end just touching the ground. Onyx sends a small transverse pulse down the rope by flicking the top.

Given parameters:

- Length of the rope: $L = 20$ m
- Linear mass density: μ
- Gravitational acceleration: $g = 9.8$ m/s^2

Task

Determine the time t required for the pulse to travel from the top of the rope to its bottom end.

Solution

The tension in a hanging rope is not uniform. At a distance y above the free lower end, the tension must support the weight of the segment below, namely

$$T(y) = \mu g y \tag{23}$$

The speed of a transverse pulse on the rope is therefore

$$v(y) = \sqrt{\frac{T}{\mu}} = \sqrt{gy} \tag{24}$$

The pulse travels from the bottom ($y = 0$) to the top ($y = L$), so the total time is

$$t = \int_0^L \frac{dy}{v(y)} = \int_0^L \frac{dy}{\sqrt{gy}} = \frac{2\sqrt{L}}{\sqrt{g}} \tag{25}$$

Giving the compact result

$$t = 2\sqrt{\frac{L}{g}} \qquad \Rightarrow \qquad t = 2\sqrt{\frac{20}{9.8}} \approx 2.9 \text{ s} \tag{26}$$

$$\boxed{t = 2\sqrt{L/g} \approx 2.9 \text{ s}} \tag{27}$$

Problem 7 — Ruling the Atom

In Princess Bubblegum's lab, a young visitor named Onyx tries to estimate the size of a hydrogen atom without solving the Schrödinger equation. He reasons that the electron is confined to a region of size r, so by the uncertainty principle its momentum is at least of order $p \sim \hbar/r$. Its kinetic energy is then $p^2/2m$, and its potential energy in the Coulomb field of the proton is $-ke^2/r$.

Task

Minimize the total energy with respect to r and thereby estimate the radius r_0 and the binding energy E_0 of the ground state of the hydrogen atom.

Solution

The total energy as a function of the confinement size is

$$E(r) \approx \frac{\hbar^2}{2mr^2} - \frac{ke^2}{r} \tag{28}$$

Minimizing with respect to r:

$$\frac{dE}{dr} = -\frac{\hbar^2}{mr^3} + \frac{ke^2}{r^2} = 0 \qquad \Rightarrow \qquad r_0 = \frac{\hbar^2}{mke^2} \tag{29}$$

This is precisely the Bohr radius:

$$r_0 = a_0 = \frac{\hbar^2}{mke^2} \approx 5.3 \times 10^{-11} \text{ m} \tag{30}$$

Substituting back into the energy:

$$E_0 = -\frac{\hbar^2}{2mr_0^2} = -\frac{mk^2e^4}{2\hbar^2} \approx -13.6 \text{ eV} \tag{31}$$

$$\boxed{r_0 \approx 0.53 \text{ Å}, \qquad E_0 \approx -13.6 \text{ eV}} \tag{32}$$

Problem 8 — The Bouncing Buoy

Off the beach of the ruined Great Mushroom War coast, Onyx moors a cylindrical buoy of mass $m = 0.5$ kg and cross-sectional area $A = 0.01$ m^2 to a dock with an ideal spring of stiffness $k = 200$ N/m. The buoy floats upright, partially submerged and intersecting the water surface. Water density is $\rho_w = 1000$ kg/m^3.

Given parameters:

- Buoy mass: $m = 0.5$ kg
- Cross-sectional area: $A = 0.01$ m^2
- Spring stiffness: $k = 200$ N/m
- Water density: $\rho_w = 1000$ kg/m^3

Task

Determine the angular frequency ω of the small vertical oscillations of the buoy about its equilibrium position.

Solution

At equilibrium the spring and buoyancy forces balance the weight. If the buoy is displaced deeper by x, two restoring contributions appear:

- The spring is stretched further, adding $-kx$;
- The submerged volume grows by Ax, adding buoyancy $-(\rho_w g A)x$.

The equation of motion is therefore

$$m\ddot{x} = -(k + \rho_w g A)\, x \tag{33}$$

which describes SHM with

$$\omega = \sqrt{\frac{k + \rho_w g A}{m}} \tag{34}$$

With the given parameters:

$$\omega = \sqrt{\frac{200 + 1000 \times 9.8 \times 0.01}{0.5}} = \sqrt{\frac{200 + 98}{0.5}} = \sqrt{596} \approx 24 \text{ rad/s}$$

$$\boxed{\omega = \sqrt{\frac{k + \rho_w g A}{m}} \approx 24 \text{ rad/s}} \tag{35}$$

Problem 9 — The Ice King's Hoist

A uniform disk-shaped pulley of mass $M = 2$ kg and radius R, looted from the Ice King's castle, hangs from the ceiling of the Tree Fort. A light string passes over it without slipping, and Onyx attaches sacks of marshmallows of masses $m_1 = 3$ kg and $m_2 = 1$ kg to the two ends. The system starts from rest.

Given parameters:

- Sack masses: $m_1 = 3$ kg, $m_2 = 1$ kg
- Pulley (uniform disk): $M = 2$ kg
- Moment of inertia of the disk: $I = MR^2/2$

Task

Determine the acceleration a of the sacks.

Solution

Let T_1 and T_2 be the tensions in the string on the heavier and lighter sides. The equations of motion are

$$m_1 g - T_1 = m_1 a \tag{36}$$
$$T_2 - m_2 g = m_2 a \tag{37}$$

The net torque on the pulley produces its angular acceleration, and because the string does not slip $\alpha = a/R$:

$$(T_1 - T_2)R = I\alpha = I\frac{a}{R} \tag{38}$$

Eliminating the tensions from (36)–(38):

$$(m_1 - m_2)g = \left(m_1 + m_2 + \frac{I}{R^2}\right) a \tag{39}$$

Using $I/R^2 = M/2$:

$$a = \frac{(m_1 - m_2)g}{m_1 + m_2 + M/2} \tag{40}$$

With the numbers ($m_1 - m_2 = 2$ kg, $m_1 + m_2 + M/2 = 5$ kg):

$$a = \frac{2 \times 9.8}{5} = 3.92 \text{ m/s}^2$$

$$\boxed{a = \frac{(m_1 - m_2)g}{m_1 + m_2 + M/2} \approx 3.9 \text{ m/s}^2} \tag{41}$$

Problem 10 — The Surprise Overhead

A white dislodged chunk of Billy's comet streaks horizontally at a constant altitude $H = 10$ km with Mach number $M = 2$ (i.e. speed $v = Mc_s$, where $c_s = 340$ m/s is the speed of sound). Onyx lies in the Grasslands with a finch, exactly on the comet's flight path.

Given parameters:

- Altitude: $H = 10$ km
- Mach number: $M = 2$
- Speed of sound: $c_s = 340$ m/s

Task

Determine the time t between the moment the comet chunk passes directly overhead and the moment Onyx first hears the sonic boom.

Solution

A source moving faster than sound creates a conical shock wave (the Mach cone) whose half-angle θ satisfies

$$\sin\theta = \frac{1}{M} \tag{42}$$

Onyx first feels the boom when the cone sweeps over his position. From the geometry, when the cone's surface passes the observer the comet chunk has moved a horizontal distance d ahead of the overhead point such that

$$\tan\theta = \frac{d}{H} \tag{43}$$

Therefore the time delay is

$$t = \frac{d}{v} = \frac{H\tan\theta}{Mc_s} = \frac{H}{Mc_s}\,\frac{\sin\theta}{\sqrt{1-\sin^2\theta}} = \frac{H}{c_s\,M\sqrt{M^2-1}} \tag{44}$$

For $M = 2$ and $H = 10$ km:

$$t = \frac{10^4}{340 \times 2 \times \sqrt{3}} = \frac{10^4}{1177.6} \approx 8.5 \text{ s}$$

$$\boxed{t = \frac{H}{c_s M \sqrt{M^2 - 1}} \approx 8.5 \text{ s}} \tag{45}$$

Answers at a glance:

P1: $t \approx 7.7$ s P2: $\omega \approx 9.5$ rad/s P3: $v_t = 50$ m/s P4: $r \approx 2.3$ m P5: $\omega \approx 30$ rad/s
P6: $t \approx 2.9$ s P7: $r_0 \approx 0.53$Å, $E_0 \approx -13.6$ eV P8: $\omega \approx 24$ rad/s P9: $a \approx 3.9$ m/s^2
P10: $t \approx 8.5$ s

proposed by Tawhid