Cosmologic — Open Category Problem Set

Friedmann Equations · Stellar Structure · Accretion Disks · CMB & Recombination · TOV Equation

Instructions. All questions require *full derivations*; a final expression without supporting working will receive no credit. You may cite standard results (Boltzmann distribution, Stefan–Boltzmann law, Kepler's third law, etc.) without re-derivation, but must state them explicitly. Physical constants carry their usual meanings throughout.

Question 1 The Friedmann Equations and Cosmological Evolution

The large-scale dynamics of the Universe are governed by the **Friedmann–Lemaître–Robertson–Walker (FLRW)** metric. The spacetime interval is

$$\mathrm{d}s^2 = -c^2\,\mathrm{d}t^2 + a(t)^2\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\,\mathrm{d}\Omega^2\right],$$

where $a(t)$ is the **scale factor** and $k \in \{-1, 0, +1\}$ is the spatial curvature. The universe is filled with a perfect fluid of energy density ρ and pressure p.

(I) By considering the **Newtonian energy equation** for a thin shell of mass m at the edge of an expanding sphere of radius $r = a(t)\,\chi$ (comoving coordinate χ), augmented with the appropriate cosmological-constant energy term $-\Lambda c^2/3$, derive the **first Friedmann equation**:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3}.$$

Identify the physical meaning of each term.

(II) Define the **density parameter** $\Omega = \rho/\rho_c$ and the **critical density** ρ_c. Show that the curvature condition reduces to

$$k = \frac{a^2H^2}{c^2}(\Omega_{\mathrm{tot}} - 1),$$

where Ω_{tot} includes matter, radiation, and a dark-energy term $\Omega_\Lambda = \Lambda c^2/(3H^2)$.

(III) The universe transitions through three epochs. For each, solve the Friedmann equation (setting $k = 0$, $\Lambda = 0$ for the first two, and $k = 0$, $\rho_{\mathrm{m}} = 0$ for the third) to find $a(t)$:

a) Radiation-dominated era: $p = \rho c^2/3$, so $\rho \propto a^{-4}$.

b) Matter-dominated era: $p = 0$, so $\rho \propto a^{-3}$.

c) Λ-dominated era: $\rho_\Lambda = \Lambda c^2/(8\pi G) = \text{const}$.

(IV) Define the **comoving Hubble radius** $d_H = c/(aH)$. Show that $\mathrm{d}(d_H)/\mathrm{d}t < 0$ requires $\ddot{a} > 0$, and hence that during inflation (which also drives $a \propto e^{Ht}$) the Hubble radius *shrinks* in comoving coordinates while growing in physical coordinates. Explain why this resolves the **horizon problem**.

Question 2 Stellar Structure and the Lane–Emden Equation

A self-gravitating, spherically symmetric star in **hydrostatic equilibrium** obeys

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{G\,M(r)\,\rho}{r^2}, \qquad \frac{\mathrm{d}M}{\mathrm{d}r} = 4\pi r^2 \rho.$$

A **polytrope** assumes the equation of state $P = K\rho^{1+1/n}$, where K and n are constants.

(I) Define a dimensionless density variable θ by $\rho = \rho_c\,\theta^n$, where ρ_c is the central density, and a dimensionless radial coordinate

$$\xi = \frac{r}{\alpha}, \qquad \alpha^2 = \frac{(n+1)K\rho_c^{1/n-1}}{4\pi G}.$$

Combine the two equations of stellar structure (eliminating $M(r)$) to derive the **Lane–Emden equation**:

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -\theta^n.$$

State the boundary conditions at $\xi = 0$.

(II) For $n = 1$, show that the Lane–Emden equation reduces to $\theta'' + (2/\xi)\theta' + \theta = 0$, verify that the solution satisfying the boundary conditions is

$$\theta_1(\xi) = \frac{\sin\xi}{\xi},$$

and find the stellar surface radius R in terms of α.

(III) Using the $n = 1$ solution, express the **total mass** M of the star as an integral over ξ and evaluate it. Write M explicitly in terms of K, ρ_c, and G.

(IV) Show by examining the behaviour of the Lane–Emden equation near the origin ($\xi \to 0$) that for *any* polytropic index n, the central boundary conditions require $\theta(0) = 1$ and $\theta'(0) = 0$. Then argue analytically (without solving numerically) that for $n = 5$ the star has *infinite radius* by examining the asymptotic behaviour $\theta \sim \xi^{-1/2}$ as $\xi \to \infty$.

Question 3 Accretion Disks and Relativistic Efficiency

Matter accreting onto a compact object forms a **thin accretion disk**. As gas spirals inward, it loses angular momentum and releases gravitational energy, which is radiated locally as blackbody radiation. General relativity requires that stable circular orbits exist only outside the **innermost stable circular orbit (ISCO)**.

(I) The accretion luminosity is limited by radiation pressure on infalling matter. For a fully ionised hydrogen plasma, derive the **Eddington luminosity**

$$L_{\rm Edd} = \frac{4\pi G M m_p c}{\sigma_T},$$

where σ_T is the Thomson cross-section and m_p is the proton mass. Define the **Eddington accretion rate** $\dot{M}_{\rm Edd}$ in terms of $L_{\rm Edd}$ and the radiative efficiency η.

(II) The Schwarzschild metric for a non-rotating black hole is

$$\mathrm{d}s^2 = -\left(1 - \frac{R_s}{r}\right) c^2\,\mathrm{d}t^2 + \left(1 - \frac{R_s}{r}\right)^{-1} \mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2.$$

A massive test particle in a circular orbit at radius r has conserved energy per unit mass $\tilde{E}$ and angular momentum per unit mass $\tilde{L}$. By forming an **effective potential** $V_{\rm eff}(r)$ from the radial geodesic equation

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 = \tilde{E}^2 - V_{\rm eff}^2(r),$$

show that the ISCO occurs at

$$r_{\rm ISCO} = \frac{6GM}{c^2} = 3R_s.$$

Hint: The ISCO is where the effective potential has a saddle point, i.e., $V'_{\rm eff} = 0$ and $V''_{\rm eff} = 0$ simultaneously.

(III) The binding energy released per unit mass as material falls from infinity to the ISCO equals $c^2(1 - \tilde{E}_{\rm ISCO})$. Using your result from (II), evaluate $\tilde{E}_{\rm ISCO}$ and hence compute the **maximum radiative efficiency** $\eta_{\rm Sch}$ for accretion onto a Schwarzschild black hole. Compare this to the $\sim 0.7\%$ efficiency of hydrogen fusion.

(IV) For a geometrically thin, optically thick disk, local blackbody emission balances the viscous dissipation rate. The viscous dissipation per unit area of the disk at radius r is (Shakura–Sunyaev):

$$D(r) = \frac{3GM\dot{M}}{8\pi r^3}\left[1 - \sqrt{\frac{r_{\rm in}}{r}}\right].$$

Setting $D(r) = \sigma T^4$, derive the **temperature profile** $T(r)$ of the disk and find the radius $r_{\rm max}$ at which the temperature peaks.

Question 4 The Cosmic Microwave Background and Recombination

As the universe cooled below $T \sim 3000\,\mathrm{K}$, free electrons combined with protons to form neutral hydrogen — the epoch of **recombination**. The photons that last scattered at this epoch form the **Cosmic Microwave Background (CMB)**.

(I) The **Saha equation** for hydrogen recombination is

$$\frac{n_e n_p}{n_H} = \left(\frac{m_e k_B T}{2\pi\hbar^2}\right)^{3/2} e^{-E_I/(k_B T)},$$

where $E_I = 13.6\,\mathrm{eV}$ is the ionisation energy of hydrogen. Define the **ionisation fraction** $x_e = n_e/(n_e + n_H)$. Assuming $n_e = n_p$ and total baryon number density $n_b = \eta\, n_\gamma \propto T^3$ (with $\eta \approx 6 \times 10^{-10}$ the baryon-to-photon ratio), re-express the Saha equation purely in terms of x_e, T, and known constants. Hence estimate the recombination temperature T_{rec} at which $x_e = 0.5$, and explain why $k_B T_{\mathrm{rec}} \ll E_I$.

(II) Before recombination, photons undergo repeated Thomson scattering and diffuse a characteristic distance known as the **Silk damping scale** λ_D. Show that the diffusion length for a photon performing a random walk of N steps of mean free path $\lambda_{\mathrm{mfp}} = 1/(n_e \sigma_T)$ over a Hubble time $t_H \sim 1/H$ is

$$\lambda_D \sim \left(\frac{c\,\lambda_{\mathrm{mfp}}}{H}\right)^{1/2}.$$

Express λ_D in terms of H, n_e, σ_T, and c, and explain qualitatively why density fluctuations on scales $\lesssim \lambda_D$ are **exponentially suppressed** in the CMB power spectrum.

(III) The **sound horizon** at recombination is the maximum comoving distance a sound wave can travel from the Big Bang to t_{rec}:

$$r_s = \int_0^{t_{\mathrm{rec}}} \frac{c_s}{a(t)}\,\mathrm{d}t,$$

where $c_s = c/\sqrt{3(1 + R_b)}$ is the photon–baryon sound speed and $R_b = 3\rho_b/(4\rho_\gamma)$. In a flat, matter-dominated universe ($a \propto t^{2/3}$, $R_b \ll 1$), show that $r_s \approx 2c/(\sqrt{3}\,H(z_{\mathrm{rec}})\,\sqrt{1 + z_{\mathrm{rec}}})$ and evaluate it numerically for $z_{\mathrm{rec}} = 1100$, $H_0 = 67\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, $\Omega_m = 0.31$. Show that this corresponds to an angular scale on the CMB sky of approximately $\theta_1 \approx 1$, consistent with the observed first acoustic peak at multipole $\ell \approx 180$.

Question 5 General Relativistic Stars and the TOV Equation

In general relativity, the equation of hydrostatic equilibrium for a spherically symmetric star is no longer the Newtonian $\mathrm{d}P/\mathrm{d}r = -GM\rho/r^2$. The full GR result is the **Tolman–**

Oppenheimer–Volkoff (TOV) equation:

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{G}{r^2}\left(\rho + \frac{P}{c^2}\right)\left(M + \frac{4\pi r^3 P}{c^2}\right)\left(1 - \frac{2GM}{rc^2}\right)^{-1}.$$

(I) Identify the **three GR correction factors** in the TOV equation relative to Newtonian hydrostatic equilibrium and give a physical interpretation of each. Show that in the non-relativistic limit ($P \ll \rho c^2$, $M \ll rc^2/(2G)$), the Newtonian result is recovered.

(II) Consider a **uniform-density star** with $\rho = \rho_0 = \text{const}$ and total mass M and radius R. The TOV equation can be integrated analytically. Define the compactness parameter $\beta = GM/(Rc^2)$. Show that the **central pressure** is

$$P_c = \rho_0 c^2 \, \frac{1 - \sqrt{1-2\beta}}{\sqrt{1-2\beta} - 3\sqrt{1-8\beta/3}}.$$

Hint: Substitute $u(r) = 1 - 2GM(r)/(rc^2)$ and look for a solution of the form $P(r) = \rho_0 c^2[f(r) - 1]$.

(III) From the central pressure formula in (II), show that $P_c \to \infty$ as $\beta \to 4/9$. Hence derive the **Buchdahl limit**:

$$R > \frac{9}{4}\,\frac{GM}{c^2} = \frac{9}{8}\,R_s,$$

and explain why no static star can be more compact than this, regardless of its equation of state.

(IV) For a neutron star, **causality** demands that the speed of sound $c_s^2 = \mathrm{d}P/\mathrm{d}\rho \le c^2$. Suppose the equation of state is maximally stiff: $P = c^2(\rho - \rho_0)$ above a fiducial nuclear-saturation density $\rho_0 = 2.3 \times 10^{17}\,\mathrm{kg\,m^{-3}}$. Using the scaling $M_{\max} \sim (c^2/G)^{3/2}(1/\rho_0)^{1/2}$, estimate the **maximum neutron star mass** in solar masses. Compare to the observed record holder PSR J0952-0607 ($M \approx 2.35\,M_\odot$) and comment on what this implies for dense-matter physics.

Numbers in [**brackets**] indicate marks. Total: [**51 marks**]. Allowed aids: standard physical constants sheet; no computer algebra systems.