Stellar Astrophysics — Mark Scheme & Solutions

Full derivations with intermediate steps · Final expressions boxed in green

Question 1 Pulsars

(I) **Minimum rotational period.** **[1]**

At the equator of the pulsar, gravity must at minimum balance the centrifugal force:

$$\frac{GMm}{R^2} \geq m\omega^2 R \quad \Longrightarrow \quad \omega_{\max} = \sqrt{\frac{GM}{R^3}}.$$

Since $P = 2\pi/\omega$, the minimum period is $P_{\min} = 2\pi/\omega_{\max}$.

$$\boxed{P_{\min} = 2\pi\sqrt{\frac{R^3}{GM}}}$$

(II) **Rate of change of the rotational period.** **[2]**

For a uniform solid sphere the moment of inertia is $I = \frac{2}{5}MR^2$. The angular frequency is $\omega = 2\pi/P$, so:

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = -\frac{2\pi\dot{P}}{P^2}.$$

The braking torque $\mathcal{N}$ supplies the angular impulse:

$$\mathcal{N} = I\,\frac{\mathrm{d}\omega}{\mathrm{d}t} = -\frac{2\pi I\,\dot{P}}{P^2}.$$

Solving for $\dot{P}$:

$$\boxed{\dot{P} = -\frac{\mathcal{N}\,P^2}{2\pi I} \quad \text{where } I = \tfrac{2}{5}MR^2}$$

Because $\mathcal{N}$ is a *braking* (negative) torque, $\dot{P} > 0$: the period grows over time.

(III) **Rate of loss of rotational kinetic energy.** **[3]**

The rotational kinetic energy is:

$$E_{\rm rot} = \tfrac{1}{2} I\omega^2 = \frac{2\pi^2 I}{P^2}.$$

Differentiating with respect to time:

$$\dot{E}_{\rm rot} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{2\pi^2 I}{P^2}\right) = -\frac{4\pi^2 I\,\dot{P}}{P^3}.$$

$$\boxed{\dot{E}_{\rm rot} = -\frac{4\pi^2 I\,\dot{P}}{P^3}}$$

The minus sign confirms that an increasing period ($\dot{P} > 0$) corresponds to a loss of rotational energy. This radiated power ($|\dot{E}|$) is called the *spin-down luminosity*.

Question 2 Jeans Instability

(I) **Jeans radius.** **[3]**

A cloud of mass M, mean particle mass μ, and number density n has mass density $\rho = n\mu$. The isothermal sound speed is $c_s^2 = k_B T/\mu$. Perturbation analysis of the fluid equations yields instability for wavelengths exceeding the Jeans length λ_J, defined by setting the oscillation frequency to zero:

$$\omega^2 = c_s^2 k^2 - 4\pi G\rho = 0 \quad\Longrightarrow\quad k_J = \sqrt{\frac{4\pi G\rho}{c_s^2}}, \quad \lambda_J = \frac{2\pi}{k_J} = \pi^{1/2}\, c_s\,\frac{1}{\sqrt{G\rho}}.$$

The Jeans *radius* is $R_J = \lambda_J/2$:

$$R_J = \frac{1}{2}\sqrt{\frac{\pi c_s^2}{G\rho}} = \frac{1}{2}\sqrt{\frac{\pi k_B T}{G\mu^2 n}}.$$

$$\boxed{R_J = \frac{1}{2}\sqrt{\frac{\pi k_B T}{G\mu^2 n}}}$$

(II) **Jeans mass.** **[2]**

Taking the Jeans mass as the mass contained within a sphere of radius R_J:

$$M_J = \frac{4\pi}{3}\,\rho\,R_J^3 = \frac{4\pi}{3}\,(n\mu)\left(\frac{1}{2}\sqrt{\frac{\pi k_B T}{G\mu^2 n}}\right)^3.$$

Expanding:

$$M_J = \frac{4\pi}{3}\,(n\mu)\cdot\frac{1}{8}\cdot\frac{\pi^{3/2}\,k_B^{3/2}\,T^{3/2}}{G^{3/2}\mu^3 n^{3/2}} = \frac{\pi^{5/2}}{6}\,\frac{k_B^{3/2}\,T^{3/2}}{G^{3/2}\,\mu^2\,n^{1/2}}.$$

$$\boxed{M_J = \frac{\pi^{5/2}}{6}\,\frac{k_B^{3/2}\,T^{3/2}}{G^{3/2}\,\mu^2\,n^{1/2}}}$$

Note: $M_J \propto T^{3/2}$, so a hotter cloud requires a larger mass to collapse.

Question 3 White Dwarfs and the Chandrasekhar Limit

(I) **Non-relativistic electron degeneracy pressure.** **[2]**

From quantum statistical mechanics, filling all states up to the Fermi momentum $p_F = \hbar(3\pi^2 n_e)^{1/3}$ gives the non-relativistic degeneracy pressure:

$$P_{\rm deg} = \frac{(3\pi^2)^{2/3}\,\hbar^2}{5\,m_e}\,n_e^{5/3}.$$

The electron number density is $n_e = \rho/(\mu_e m_p)$, where μ_e is the mean molecular weight per electron and m_p is the proton mass.

$$\boxed{P_{\rm deg} = \frac{(3\pi^2)^{2/3}\,\hbar^2}{5\,m_e}\,n_e^{5/3}, \qquad n_e = \frac{3M}{4\pi R^3\,\mu_e m_p}}$$

(II) **White dwarf mass–radius relation.** **[2]**

Gravitational pressure at the centre scales as $P_{\rm grav} \sim GM^2/R^4$. Setting $P_{\rm deg} \sim P_{\rm grav}$ with $n_e \propto M/R^3$:

$$\frac{\hbar^2}{m_e}\left(\frac{M}{m_p R^3}\right)^{5/3} \sim \frac{GM^2}{R^4}.$$

Rearranging (the R dependence: $R^{-5} \sim R^{-4}$ gives $R^{-1} \sim M^{1/3}$):

$$R \propto M^{-1/3} \quad \Longrightarrow \quad R\,M^{1/3} = \text{const.}$$

$$\boxed{R = \frac{(18\pi)^{2/3}}{10}\,\frac{\hbar^2}{G\,m_e\,(\mu_e m_p)^{5/3}}\,M^{-1/3}}$$

More massive white dwarfs are *smaller*: a hallmark of degenerate matter.

(III) **Chandrasekhar mass limit.** **[2]**

In the ultra-relativistic limit, $P_{\rm rel} \propto \hbar c\, n_e^{4/3}$. Balancing with gravity:

$$\hbar c \left(\frac{M}{m_p R^3}\right)^{4/3} \sim \frac{GM^2}{R^4}.$$

Both sides scale as R^{-4}, so there is a unique mass for which this balance holds. Solving for M:

$$\boxed{M_{\rm Ch} = \frac{5.87}{\mu_e^2}\,M_\odot \approx \left(\frac{\hbar c}{G}\right)^{3/2} \frac{1}{(\mu_e\,m_p)^2}}$$

For $\mu_e = 2$ (fully ionised carbon/oxygen), $M_{\rm Ch} \approx 1.4\,M_\odot$. Above this limit, degeneracy pressure cannot support the star, triggering collapse or a Type Ia supernova.

Question 4 Gravitational Waves from Binary Pulsars

(I) **Total orbital energy.** **[1]**

For two point masses m_1, m_2 in a circular orbit of separation a, the virial theorem gives $E_{\rm kin} = -\frac{1}{2}E_{\rm pot}$, so:

$$E_{\rm orb} = E_{\rm kin} + E_{\rm pot} = \frac{1}{2}E_{\rm pot} = -\frac{Gm_1m_2}{2a}.$$

$$\boxed{E_{\rm orb} = -\frac{Gm_1m_2}{2a}}$$

(II) **Gravitational wave luminosity (Peters formula).** **[3]**

General relativity shows that a binary system radiates gravitational waves at the rate (Peters 1964):

$$\mathcal{L}_{\rm GW} = \frac{32}{5}\frac{G^4}{c^5}\frac{(m_1 m_2)^2(m_1+m_2)}{a^5}.$$

This can be written compactly using the *chirp mass* $\mathcal{M} = (m_1 m_2)^{3/5}/(m_1+m_2)^{1/5}$:

$$\mathcal{L}_{\rm GW} = \frac{32}{5}\frac{c^5}{G}\left(\frac{G\mathcal{M}}{c^2 a}\right)^5.$$

$$\boxed{\mathcal{L}_{\rm GW} = \frac{32G^4}{5c^5}\frac{(m_1 m_2)^2(m_1+m_2)}{a^5}}$$

(III) **Rate of change of orbital period.** **[2]**

From Kepler's third law, $P_b^2 = 4\pi^2 a^3/[G(m_1+m_2)]$, so $a \propto P_b^{2/3}$. Differentiating $E_{\rm orb}$ and equating to $-\mathcal{L}_{\rm GW}$:

$$\dot{E}_{\rm orb} = \frac{Gm_1 m_2}{2a^2}\,\dot{a} = -\mathcal{L}_{\rm GW}.$$

Since $\dot{P}_b = \frac{3}{2}(P_b/a)\dot{a}$, substituting and simplifying:

$$\boxed{\dot{P}_b = -\frac{192\pi}{5}\left(\frac{G}{c^3}\right)^{5/3}\frac{m_1 m_2}{(m_1+m_2)^{1/3}}\left(\frac{2\pi}{P_b}\right)^{5/3}}$$

$\dot{P}_b < 0$: the orbit shrinks as energy is radiated away. This was confirmed observationally by Hulse & Taylor (Nobel Prize, 1993).

Question 5 Black Holes and Hawking Radiation

(I) **Schwarzschild radius.** **[1]**

The escape velocity at radius R equals the speed of light when:

$$\frac{1}{2}mc^2 = \frac{GMm}{R} \quad\Longrightarrow\quad R_s = \frac{2GM}{c^2}.$$

(A full GR derivation yields the same expression from the vanishing of the Schwarzschild metric g_{tt}.)

$$\boxed{R_s = \frac{2GM}{c^2}}$$

(II) **Hawking temperature.** **[1]**

Quantum field theory in curved spacetime shows that the event horizon produces thermal radiation at the Hawking temperature (Hawking 1974):

$$\boxed{T_H = \frac{\hbar\, c^3}{8\pi\, G\, M\, k_B}}$$

$T_H \propto 1/M$: stellar-mass black holes are absurdly cold ($T_H \ll 10^{-7}\,\mathrm{K}$), while microscopic black holes would be extremely hot.

(III) **Rate of mass loss and evaporation lifetime.** **[3]**

Treating the black hole as a blackbody radiator of radius R_s at temperature T_H, the Stefan–Boltzmann law gives the luminosity:

$$\mathcal{L} = \sigma T_H^4 \cdot 4\pi R_s^2.$$

Substituting T_H and $R_s = 2GM/c^2$ and using $\sigma = \pi^2 k_B^4/(60\hbar^3 c^2)$:

$$\mathcal{L} = \frac{\hbar\, c^6}{15360\,\pi\, G^2\, M^2}.$$

Since the mass–energy loss rate is $\mathcal{L} = -\dot{M}c^2$:

$$\frac{\mathrm{d}M}{\mathrm{d}t} = -\frac{\hbar\, c^4}{15360\,\pi\, G^2\, M^2}.$$

Separating variables and integrating from M_0 to 0:

$$\int_0^\tau \mathrm{d}t = -\frac{15360\,\pi\, G^2}{\hbar\, c^4}\int_{M_0}^0 M^2\,\mathrm{d}M = \frac{15360\,\pi\, G^2}{\hbar\, c^4}\,\frac{M_0^3}{3}.$$

$$\boxed{\frac{\mathrm{d}M}{\mathrm{d}t} = -\frac{\hbar\, c^4}{15360\,\pi\, G^2\, M^2}}$$

$$\boxed{\tau = \frac{5120\,\pi\,G^2\,M_0^3}{\hbar\,c^4}}$$

For a solar-mass black hole, $\tau \approx 2 \times 10^{74}\,\mathrm{yr}$ — far exceeding the current age of the universe. A black hole with $M_0 \sim 10^{11}\,\mathrm{kg}$ (asteroid-mass) would evaporate in $\sim 10^{10}\,\mathrm{yr}$.

End of mark scheme. Total: [**28 marks**]. Expressions in **green boxes** are the final required answers. Steps in **tan boxes** show the required working.