Stellar Astrophysics — Problem Set

Pulsars · Jeans Collapse · White Dwarfs · Binary Systems · Black Holes

Question 1 Pulsars

When a massive star exhausts its nuclear fuel, it undergoes a supernova explosion, leaving behind a core composed almost entirely of neutrons — a **neutron star**. A rotating neutron star possesses a strong magnetic field and emits beams of electromagnetic radiation along its magnetic poles; when these beams sweep past Earth, we observe periodic pulses and call the object a **pulsar**. Because angular momentum is conserved during the collapse, pulsars rotate extremely rapidly. Over time, a pulsar loses rotational energy as a relativistic wind of particles streams outward.

(I) Write an expression for the **minimum rotational period** of a pulsar of mass M and radius R, below which centrifugal forces at the equator would exceed self-gravity.

(II) Treating the neutron star as a **uniform solid sphere**, write an expression for the **rate of change of the rotational period** $\dot{P}$ in terms of the moment of inertia I, the period P, and the braking torque $\mathcal{N}$.

(III) Write an expression for the **rate of loss of rotational kinetic energy** $\dot{E}$ as a function of I, P, and $\dot{P}$.

Question 2 Jeans Instability

A self-gravitating gas cloud is stable against collapse when thermal pressure exceeds gravitational attraction. The critical length scale below which thermal energy can no longer support the cloud is the **Jeans radius**, and the corresponding critical mass is the **Jeans mass**. Clouds exceeding the Jeans mass are unstable and will collapse to form stars.

(I) For a gas cloud of mass M and temperature T with mean particle mass μ and number density n, write an expression for the **Jeans radius** R_J.

(II) Write an expression for the **Jeans mass** M_J in terms of the temperature T, mean particle mass μ, and number density n.

Question 3 White Dwarfs and the Chandrasekhar Limit

When a low- to intermediate-mass star depletes its nuclear fuel, it sheds its outer layers and leaves behind a **white dwarf** — a dense, Earth-sized remnant supported against collapse by **electron degeneracy pressure** rather than thermal pressure. There exists a maximum mass beyond which degeneracy pressure can no longer balance gravity; this is the **Chandrasekhar limit**, $M_{\mathrm{Ch}} \approx 1.4\,M_\odot$.

(I) For a white dwarf of mass M and radius R, write an expression for the **electron degeneracy pressure** P_{deg} in the non-relativistic limit.

(II) By balancing degeneracy pressure against gravitational pressure, write an expression for the **mass–radius relation** of a white dwarf.

(III) Write an expression for the **Chandrasekhar mass limit** M_{Ch} in terms of fundamental constants and the mean molecular weight per electron μ_e.

Question 4 Gravitational Waves from Binary Pulsars

A **binary pulsar** system consists of two neutron stars in mutual orbit. General relativity predicts that such an accelerating mass distribution radiates energy in the form of **gravitational waves**. As energy is radiated away, the orbital separation shrinks and the orbital period decreases — a prediction famously confirmed by Hulse and Taylor (1974), earning them the 1993 Nobel Prize in Physics.

(I) For a binary system with component masses m_1 and m_2 in a circular orbit of separation a, write an expression for the **total orbital energy** E_{orb}.

(II) Write an expression for the **gravitational wave luminosity** $\mathcal{L}_{\mathrm{GW}}$ (Peters formula) in terms of the masses and orbital separation.

(III) Hence, write an expression for the **rate of change of the orbital period** $\dot{P}_b$ due to gravitational wave emission.

Question 5 Black Holes and Hawking Radiation

When a stellar remnant exceeds approximately $3\,M_\odot$, no known pressure can halt its gravitational collapse, and a **black hole** forms. The boundary from within which nothing — not even light — can escape is the **event horizon**, located at the Schwarzschild radius R_s. A remarkable result of quantum field theory in curved spacetime (Hawking, 1974) is that black holes are not perfectly black: they radiate thermally at a temperature inversely proportional to their mass, slowly losing mass in a process known as **Hawking radiation**.

(I) Write an expression for the **Schwarzschild radius** R_s of a black hole of mass M.

(II) Write an expression for the **Hawking temperature** T_H of a Schwarzschild black hole in terms of M and fundamental constants.

(III) Write an expression for the **rate of mass loss** $\dot{M}$ due to Hawking radiation, and hence derive an expression for the **evaporation lifetime** τ of the black hole.

Numbers in [**brackets**] indicate marks. Total: [**28 marks**].