Central Force Motion
Class 2
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Key Topics and Concepts

1. Central Force Definition A central force is directed toward or away from a fixed point, with magnitude depending only on distance r: $\vec{F}(r) = F(r)\,\hat{r}$. Examples include gravitational, electrostatic, and spring forces.

2. Polar Coordinates and Kinematics Position, velocity, and acceleration in polar coordinates:

$$\vec{r} = r\hat{r}, \qquad \vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}, \qquad \vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$$

3. Conservation Laws Central forces exert zero torque about the center, so angular momentum $\vec{L} = m\vec{r} \times \vec{v}$ is conserved (both magnitude and direction). Total mechanical energy is also conserved for conservative central forces.

4. Equations of Motion Radial: $m(\ddot{r} - r\dot{\theta}^2) = F(r)$. Angular: $m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0$, which is equivalent to $dL/dt = 0$.

5. Effective Potential Using $l = mr^2\dot{\theta} = \text{constant}$, the radial problem reduces to one-dimensional motion:

$$E = \tfrac{1}{2}m\dot{r}^2 + V_{\text{eff}}(r), \qquad V_{\text{eff}}(r) = \frac{l^2}{2mr^2} + V(r)$$

6. Binet's Orbit Equation With $u = 1/r$, the orbit equation becomes:

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{l^2}\,F\left(\frac{1}{u}\right)\frac{1}{u^2}$$

7. Inverse-Square Law Solution For $F(r) = -k/r^2$, the orbit is a conic section:

$$r(\theta) = \frac{l^2/(mk)}{1 + e\cos(\theta - \theta_0)}$$

Orbit type depends on eccentricity e: $e = 0$ (circle), $0 < e < 1$ (ellipse), $e = 1$ (parabola), $e > 1$ (hyperbola).

8. Orbit Classification by Energy $E < 0$: bound (ellipse/circle), $E = 0$: marginally unbound (parabola), $E > 0$: unbound (hyperbola).

9. Small Radial Oscillations Near a circular orbit r_0, perturbations produce simple harmonic motion:

$$\Omega_r^2 = \frac{1}{m} \left. \frac{d^2 V_{\text{eff}}}{dr^2} \right|_{r=r_0}$$

1 Central Force Motion — Lecture Notes

1.1 Introduction to Central Force Motion

A **central force** is defined as a force whose direction is always toward or away from a fixed point (the center). Common examples include gravitational force and electrostatic force, acting in planetary or atomic systems. The force can be either **attractive** (e.g., gravity) or **repulsive** (e.g., electrostatic repulsion). Central forces have **magnitude dependent on distance** from the fixed point, often proportional to inverse powers of radius r.

> **Central Force Definition:** A force $\vec{F}$ is central if it is directed along the line joining the particle to a fixed point O and its magnitude depends only on the distance r from O:
>
> $$\vec{F}(r) = F(r)\,\hat{r}$$

1.2 Nature of Central Forces and Conservativeness

The electrostatic force and gravitational force are **conservative forces**. The spring force is another example of a conservative central force, generally proportional to displacement ($F = -kr$). Trajectories under central forces can be conic sections (parabola, hyperbola, ellipse) based on energy and force parameters. The force acts on a **particle of mass** m located at a distance r from the fixed center.

> **Common Central Forces:**
>
> $$\begin{aligned} \text{Gravitational:} \quad & \vec{F} = -\frac{GM_1M_2}{r^2}\,\hat{r} \quad \text{(attractive)} \\ \text{Electrostatic:} \quad & \vec{F} = k\frac{q_1q_2}{r^2}\,\hat{r} \quad \text{(attractive or repulsive)} \\ \text{Spring:} \quad & \vec{F} = -kr\,\hat{r} \quad \text{(restoring)} \end{aligned}$$

1.3 Conditions for Central Force Motion

For motion under a central force:

- The force field must be **conservative** (total mechanical energy conserved).
- **Angular momentum is conserved** in the absence of external torque.
- Motion is confined to a **plane**, allowing usage of **polar coordinates** (r, θ).

1.4 Motion Description in Polar Coordinates

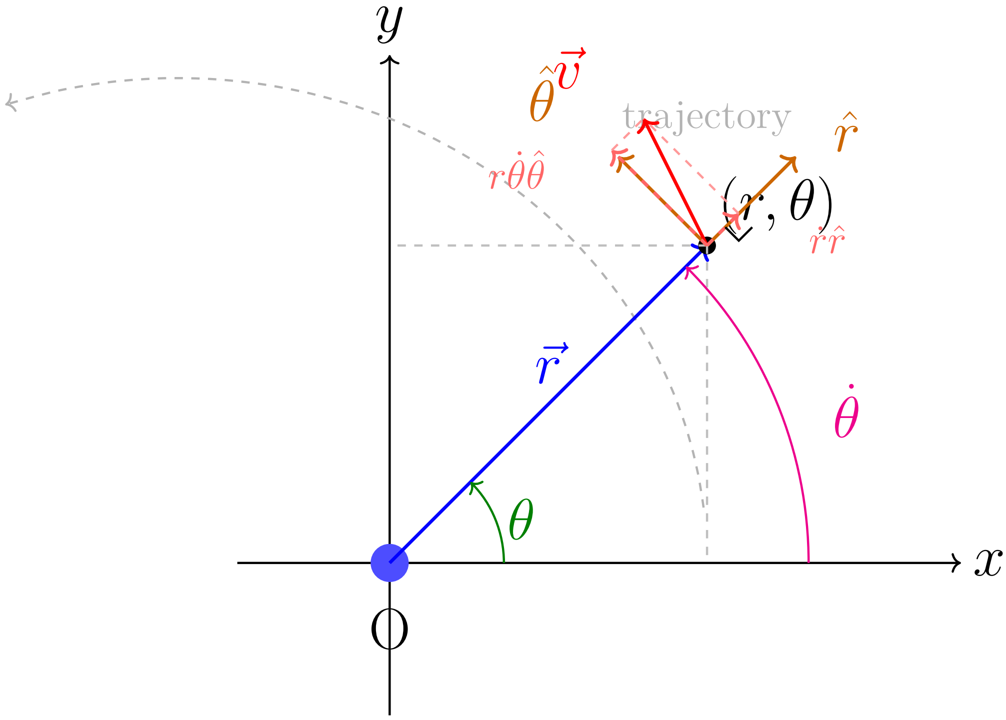

Figure 1: Polar coordinate system (r, θ) showing the position vector $\vec{r}$, unit vectors $\hat{r}$ and $\hat{\theta}$, and velocity decomposition $\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$. The central body is at focus O, and the trajectory is a conic section.

Position, velocity, and acceleration decomposed into radial and angular components:

$$\begin{aligned} \text{Position:} \quad & \vec{r} = r\hat{r} \\ \text{Velocity:} \quad & \vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \\ \text{Acceleration:} \quad & \vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta} \end{aligned}$$

The radial component corresponds to movement toward or away from the fixed point; the angular component corresponds to rotation about the center. For **uniform circular motion** (constant radius), radial velocity $\dot{r} = 0$.

1.5 Angular Momentum Conservation

The angular momentum $\vec{L}$ is given by:

$$\vec{L} = m\,\vec{r} \times \vec{v}$$

Central force implies zero external torque:

$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} = 0$$

Hence, **angular momentum magnitude and direction are constant**. This constrains the particle's motion to a plane and dictates conserved quantities related to orbits.

The velocity can also be expressed in terms of angular velocity $\vec{\omega}$:

$$\vec{v} = \dot{r}\hat{r} + \vec{\omega} \times \vec{r}$$

For a central force, the angular momentum magnitude is:

$$l = mr^2\dot{\theta} = \text{constant}$$

1.6 Equation of Motion for Particle Under Central Force

Radial equation:

$$m(\ddot{r} - r\dot{\theta}^2) = F(r)$$

Angular equation:

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0$$

The angular equation simplifies using $l = mr^2\dot{\theta}$:

$$\frac{d}{dt}(mr^2\dot{\theta}) = 0 \quad \Rightarrow \quad l = \text{constant}$$

Total mechanical energy:

$$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r)$$

where $V(r)$ is the potential energy related to $F(r) = -dV/dr$.

1.7 Energy and Effective Potential in Orbital Motion

Leveraging conserved angular momentum $l = mr^2\dot{\theta}$, the total energy becomes:

$$E = \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} + V(r)$$

The term $\dfrac{l^2}{2mr^2}$ acts as an **effective potential barrier** due to angular momentum, preventing collapse into the center. Radial motion is analyzed as one-dimensional motion in this effective potential:

$$V_{\text{eff}}(r) = \frac{l^2}{2mr^2} + V(r)$$

The radial velocity is:

$$\dot{r} = \sqrt{\frac{2}{m}\big(E - V_{\text{eff}}(r)\big)}$$

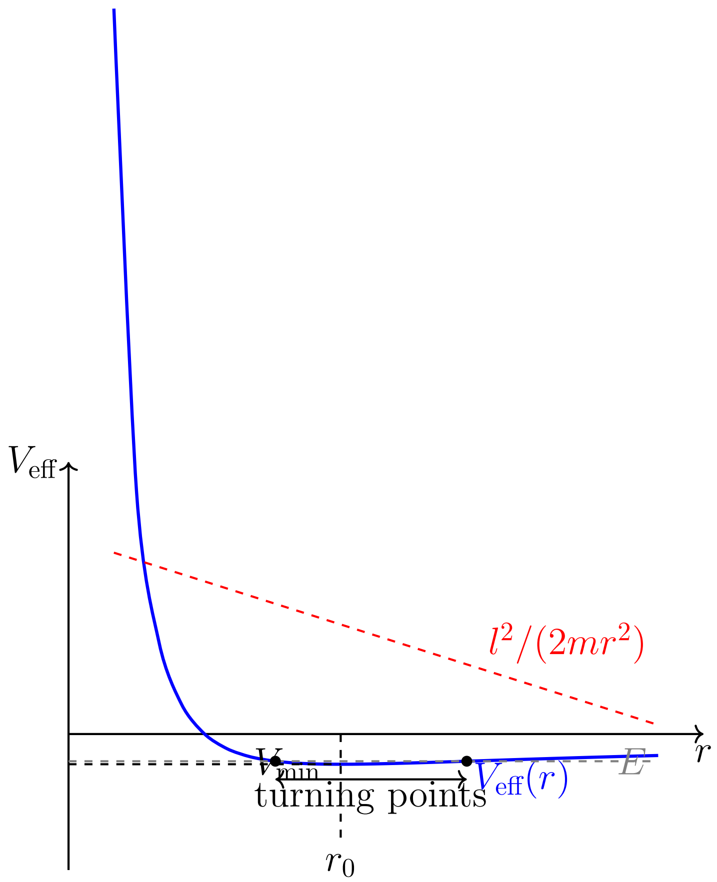

Figure 2: Effective potential $V_{\text{eff}}(r) = l^2/(2mr^2) + V(r)$ for an attractive inverse-square force. The minimum occurs at r_0 where $V'_{\text{eff}}(r_0) = 0$, corresponding to a circular orbit. A horizontal line at energy E intersects the curve at the turning points.

1.8 Orbit Equation and Binet's Formula

Introduce the substitution $u = 1/r$ to simplify the orbit equation. Using $l = mr^2\dot{\theta}$, the radial equation transforms into a differential equation in θ:

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{l^2}\,F\left(\frac{1}{u}\right)\frac{1}{u^2}$$

This is the **Binet formula** — a linear second-order ODE whose right-hand side is determined by the specific force law. The solution gives the shape of the orbit (conic sections).

1.9 Solution of the Orbit Equation for Inverse-Square Law

For gravitational and electrostatic forces, $F(r) = -k/r^2$ where $k > 0$ for attraction. Substituting into Binet's formula:

$$\frac{d^2u}{d\theta^2} + u = \frac{km}{l^2}$$

The general solution is:

$$u(\theta) = \frac{km}{l^2}\left[1 + e\cos(\theta - \theta_0)\right]$$

or equivalently in terms of r:

$$r(\theta) = \frac{l^2/(km)}{1 + e\cos(\theta - \theta_0)}$$

where e is the eccentricity, determined by initial conditions and energy.

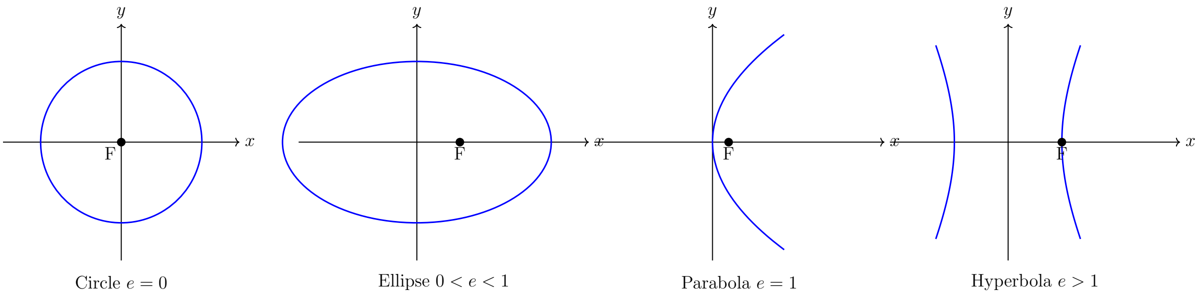

Figure 3: Conic sections as orbits under the inverse-square central force. The central body is at the focus F. The eccentricity e determines the orbit shape.

The eccentricity is related to the energy and angular momentum:

$$e = \sqrt{1 + \frac{2El^2}{mk^2}}$$

Corresponding orbit shapes depending on e:

- $e = 0$: Circle ($E = -\frac{mk^2}{2l^2}$)
- $0 < e < 1$: Ellipse ($-\frac{mk^2}{2l^2} < E < 0$)
- $e = 1$: Parabola ($E = 0$)
- $e > 1$: Hyperbola ($E > 0$)

1.10 Properties of Orbits under Central Forces

Key Properties:

- All orbits lie in a plane.

- Orbits are **conic sections** focused on the fixed force center.
- Type of orbit depends on total energy and angular momentum.
- **Attractive forces** yield closed orbits (ellipses or circles) or open trajectories (parabolas, hyperbolas) based on energy.

1.11 Orbit Classification Based on Energy

Energy	Orbit Type	Characteristics
$E < 0$	Ellipse or Circle	Bound orbit, closed trajectory
$E = 0$	Parabola	Marginally unbound, escape velocity
$E > 0$	Hyperbola	Unbound orbit, scattering/escape

For near-circular orbits, small perturbations cause simple harmonic oscillations about the mean radius.

1.12 Small Oscillations Around Circular Orbits

Consider a circular orbit radius r_0. Apply a small perturbation:

$$r = r_0 + \Delta r, \qquad \frac{\Delta r}{r_0} \ll 1$$

The perturbed motion behaves like a **simple harmonic oscillator** with angular frequency:

$$\Omega_r^2 = \frac{1}{m} \frac{d^2 V_{\text{eff}}}{dr^2}\bigg|_{r=r_0}$$

For the inverse-square law $V(r) = -k/r$, the effective potential is:

$$V_{\text{eff}}(r) = \frac{l^2}{2mr^2} - \frac{k}{r}$$

At $r = r_0 = l^2/(mk)$:

$$\frac{d^2 V_{\text{eff}}}{dr^2}\bigg|_{r=r_0} = \frac{mk^4}{l^6} > 0$$

Hence $\Omega_r = \frac{mk^2}{l^3} = \dot{\theta}(r_0)$, i.e., the radial oscillation frequency equals the orbital frequency — the orbit closes after one period (a hallmark of the inverse-square law).

1.13 Conditions for Closed Orbits and Minimum Approach Distance

Closed orbit condition requires negative total energy ($E < 0$). The **minimum distance** between two masses during orbit, r_{min}, corresponds to the point where radial velocity is zero:

$$\dot{r} = 0 \quad \Rightarrow \quad E = V_{\text{eff}}(r_{\min})$$

The conservation of energy and angular momentum equations can be solved for $r_{\min}$. The minimum distance is symmetric with the maximum distance in bounded orbits.

1.14 Summary of Conservation Laws and Their Applications

Angular momentum conservation about the force center implies motion remains planar with fixed $l = mr^2\dot{\theta}$.
Energy conservation constrains orbit shapes and allowable radial distances.
Central force problems are solved using polar coordinates with variables $r(t)$ and $\theta(t)$.
Examples include planetary orbits, satellite dynamics, and atomic-scale force interactions.

1.15 Frequency of Small Radial Oscillations

The frequency of small oscillations about the equilibrium orbit radius is:

$$\Omega_r = \sqrt{\frac{1}{m} \left. \frac{d^2 V_{\text{eff}}}{dr^2} \right|_{r=r_0}}$$

This oscillation frequency determines the stability and response of the orbit to perturbations. The angular frequency Ω_θ relates to the orbital motion around the center. When Ω_r/Ω_θ is rational, the orbit is closed; otherwise it is a precessing ellipse.

2 Problems

2.0.1 Problem 1 (Elliptic Orbit)

Problem

A satellite of mass m_s is in an elliptical orbit around a planet of mass m_p which is located at one focus of the ellipse. The satellite has a velocity v_a at the distance r_a when it is furthest from the planet (aphelion).

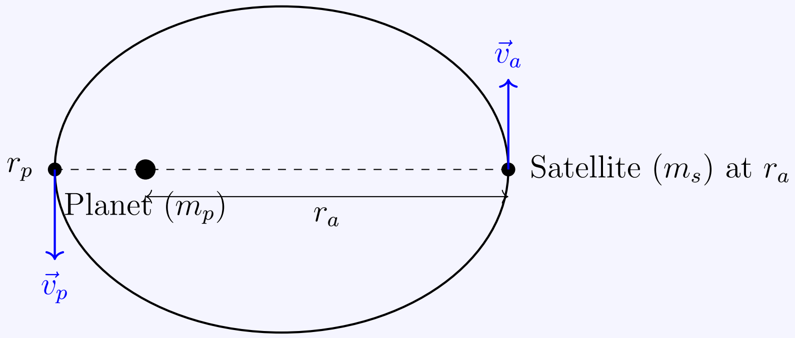

Figure 4: Elliptical orbit of a satellite around a planet.

a) What is the magnitude of the velocity v_p of the satellite when it is closest to the planet? Express your answer in terms of m_s, m_p, G, v_a, and r_a as needed.

b) What is the distance of nearest approach r_p?

c) If the satellite were in a circular orbit of radius $r_0 = r_p$, is its velocity v_0 greater than, equal to, or less than the velocity v_p of the original elliptic orbit? Justify your answer.

Solution

Solution:
(a) From conservation of angular momentum about the planet:

$$m_s v_a r_a = m_s v_p r_p \quad \Rightarrow \quad v_p = v_a \frac{r_a}{r_p}$$

From conservation of energy:

$$\frac{1}{2} m_s v_a^2 - \frac{G m_p m_s}{r_a} = \frac{1}{2} m_s v_p^2 - \frac{G m_p m_s}{r_p}$$

Substitute $v_p = v_a r_a / r_p$:

$$\frac{1}{2} v_a^2 - \frac{G m_p}{r_a} = \frac{1}{2} v_a^2 \frac{r_a^2}{r_p^2} - \frac{G m_p}{r_p}$$

$$G m_p \left(\frac{1}{r_p} - \frac{1}{r_a} \right) = \frac{1}{2} v_a^2 \left(\frac{r_a^2}{r_p^2} - 1 \right)$$

Using $r_a^2/r_p^2 - 1 = (r_a - r_p)(r_a + r_p)/r_p^2$ and $1/r_p - 1/r_a = (r_a - r_p)/(r_a r_p)$:

$$\frac{Gm_p(r_a - r_p)}{r_a r_p} = \frac{1}{2}v_a^2 \frac{(r_a - r_p)(r_a + r_p)}{r_p^2}$$
$$\frac{2Gm_p}{r_a} = v_a^2 \frac{r_a + r_p}{r_p}$$

Thus:

$$r_p = \frac{2Gm_p}{v_a^2} - r_a$$

(b) The distance of nearest approach:

$$\boxed{r_p = \frac{2Gm_p}{v_a^2} - r_a}$$

(c) For a circular orbit of radius $r_0 = r_p$:

$$v_0 = \sqrt{\frac{Gm_p}{r_p}}$$

For the elliptical orbit, using angular momentum $v_p = v_a r_a / r_p$ and energy conservation:

$$v_p^2 = \frac{2Gm_p}{r_p} - \frac{2Gm_p}{r_a + r_p}$$

Comparing $v_0^2 = Gm_p/r_p$ with v_p^2, we have $v_p > v_0$ because the satellite at perihelion of an ellipse moves faster than the circular speed at that radius (it is closer to the focus, and must overcome the gravitational pull to climb back to aphelion). This follows from the vis-viva equation: $v_p^2 = GM(2/r_p - 1/a)$ where $a > r_p$, so $v_p > \sqrt{GM/r_p} = v_0$.

2.0.2 Problem 2 (Planetary Orbits — Comet Encke)

Problem

Comet Encke was discovered in 1786 by Pierre Méchain, and in 1822 Johann Encke determined that its period was 3.3 years. It was photographed in 1913 at the aphelion distance, $r_a = 6.1 \times 10^{11}$ m (furthest distance from the Sun), by the telescope at Mt. Wilson. The distance of closest approach to the Sun, perihelion, is $r_p = 5.1 \times 10^{10}$ m.
The universal gravitational constant is $G = 6.7 \times 10^{-11}$ N·m^2·kg^{-2}, and the mass of the Sun is $m_\odot = 2.0 \times 10^{30}$ kg.

a) Explain why angular momentum is conserved about the focal point and then write down an equation for the conservation of angular momentum between aphelion and perihelion.

b) Explain why mechanical energy is conserved and then write down an equation for conservation of energy between aphelion and perihelion.

c) Use conservation of energy and angular momentum to find the speeds at perihelion (v_p) and aphelion (v_a).

Solution

Solution:
(a) The gravitational force on the comet is always directed toward the Sun (a central force). Therefore the torque about the Sun is $\vec{\tau} = \vec{r} \times \vec{F} = 0$, so angular momentum $\vec{L}$ is conserved. At the apsides, velocity is perpendicular to the radius vector, hence:

$$L = mv_a r_a = mv_p r_p \quad \Rightarrow \quad v_p = v_a \frac{r_a}{r_p}$$

(b) Gravity is a conservative force, so total mechanical energy is conserved. With zero potential at infinity:

$$\frac{1}{2}mv_a^2 - \frac{Gm_\odot m}{r_a} = \frac{1}{2}mv_p^2 - \frac{Gm_\odot m}{r_p}$$

(c) From angular momentum: $v_p = v_a\, r_a/r_p$. Substituting into energy conservation:

$$\begin{aligned}
\frac{1}{2}v_a^2 - \frac{Gm_\odot}{r_a} &= \frac{1}{2}v_a^2\frac{r_a^2}{r_p^2} - \frac{Gm_\odot}{r_p} \\
\frac{1}{2}v_a^2\left(1 - \frac{r_a^2}{r_p^2}\right) &= Gm_\odot\left(\frac{1}{r_a} - \frac{1}{r_p}\right) \\
v_a^2\frac{r_p^2 - r_a^2}{2r_p^2} &= Gm_\odot\frac{r_p - r_a}{r_a r_p} \\
v_a^2 &= \frac{2Gm_\odot r_p}{r_a(r_a + r_p)}
\end{aligned}$$

Thus:

$$\boxed{v_a = \sqrt{\frac{2Gm_\odot r_p}{r_a(r_a + r_p)}}}, \qquad \boxed{v_p = v_a\frac{r_a}{r_p} = \sqrt{\frac{2Gm_\odot r_a}{r_p(r_a + r_p)}}}$$

Numerically:

$$v_a = \sqrt{\frac{2(6.7 \times 10^{-11})(2.0 \times 10^{30})(5.1 \times 10^{10})}{(6.1 \times 10^{11})(6.61 \times 10^{11})}} \approx 9.1 \times 10^3 \text{ m/s}$$

$$v_p = \sqrt{\frac{2(6.7 \times 10^{-11})(2.0 \times 10^{30})(6.1 \times 10^{11})}{(5.1 \times 10^{10})(6.61 \times 10^{11})}} \approx 1.09 \times 10^5 \text{ m/s}$$

2.0.3 Problem 3 (Central Force Motion Theory)

Problem

A planet of mass m is orbiting a star of mass M ($M \gg m$) under a central force $\vec{F}(r)$. The planet's position in polar coordinates (r, θ) satisfies:

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}, \qquad \vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$$

Consider the central force to be gravitational for parts (e–h).

(a) Obtain the radial equation of motion $m(\ddot{r} - r\dot{\theta}^2) = F(r)$.

(b) Show that the magnitude of angular momentum L is constant.

(c) Obtain the total energy E of the planet. (Zero potential at large r)

(d) State the effective potential $V_{\text{eff}}(r)$ such that $E = \frac{1}{2}m\dot{r}^2 + V_{\text{eff}}(r)$.

(e) Sketch $V_{\text{eff}}(r)$ as a function of r.

(f) Comment on allowed regions and orbit shape for $E > 0$ and $E < 0$.

(g) Obtain the circular orbit radius r_0 in terms of L, m, etc.

(h) For $r = r_0 + x$ with $x/r_0 \ll 1$, show the planet oscillates SHM and find the period T.

(i) For $F(r) = -cr^n$ ($c > 0$), for what n is a stable circular orbit possible?

Solution

Solution:

(a) From Newton's 2nd law in polar form, the radial component gives:

$$m(\ddot{r} - r\dot{\theta}^2) = F(r)$$

(b) The angular equation is $m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0$, which is:

$$\frac{d}{dt}(mr^2\dot{\theta}) = 0 \quad \Rightarrow \quad L = mr^2\dot{\theta} = \text{constant}$$

(c) Total mechanical energy:

$$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{GMm}{r}$$

(d) Using $L = mr^2\dot{\theta}$:

$$E = \frac{1}{2}m\dot{r}^2 + \underbrace{\frac{L^2}{2mr^2} - \frac{GMm}{r}}_{V_{\text{eff}}(r)}$$

(e) Sketch: $V_{\text{eff}}(r)$ diverges as $r \to 0$ (centrifugal barrier), approaches 0^- as $r \to \infty$, with a minimum at $r_0 = L^2/(GMm^2)$.
(f) $E > 0$: Unbound hyperbolic orbit (single turning point). $E < 0$: Bound elliptical orbit (two turning points).
(g) For circular orbit, $dV_{\text{eff}}/dr = 0$:

$$-\frac{L^2}{mr_0^3} + \frac{GMm}{r_0^2} = 0 \quad \Rightarrow \quad r_0 = \frac{L^2}{GMm^2}$$

(h) For $r = r_0 + x$, expanding V_{eff} to 2nd order:

$$V_{\text{eff}}(r) \approx V_{\text{eff}}(r_0) + \frac{1}{2}V''_{\text{eff}}(r_0)\, x^2$$

$V''_{\text{eff}}(r_0) = 3GMm/r_0^3 > 0$, giving SHM with:

$$\Omega_r = \sqrt{\frac{V''_{\text{eff}}(r_0)}{m}} = \sqrt{\frac{GM}{r_0^3}}, \qquad T = \frac{2\pi}{\Omega_r} = 2\pi\sqrt{\frac{r_0^3}{GM}}$$

(i) For $F(r) = -cr^n$, stable circular orbits exist when $V''_{\text{eff}}(r_0) > 0$. The effective potential is $V_{\text{eff}} = L^2/(2mr^2) + cr^{n+1}/(n+1)$ (for $n \neq -1$). The condition $V''_{\text{eff}}(r_0) > 0$ yields:

$$\frac{3L^2}{mr_0^4} + cnr_0^{n-1} > 0 \quad \Rightarrow \quad n > -3$$

For $n = -2$ (inverse-square), $n = -1$ (logarithmic), $n = 1$ (harmonic oscillator), etc., stable orbits exist when $n > -3$.

2.0.4 Problem 4 (Two-Body Gravitational Interaction)

Problem

Two identical masses m separated by a distance ℓ are given initial transverse velocities v_0 in opposite directions as shown. The masses interact only through universal gravitation.

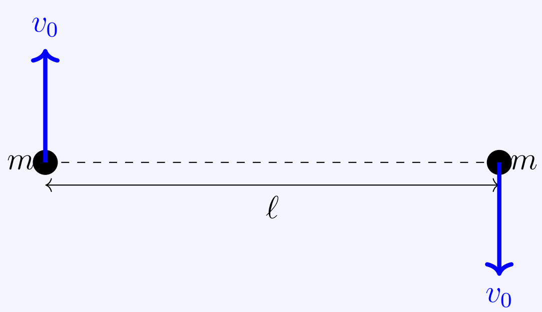

Figure 5: Two-body system with initial velocities v_0.

a. Under what conditions will the masses eventually collide?

b. Under what conditions will the masses follow circular orbits of diameter ℓ?

c. Under what conditions will the masses follow closed (bound) orbits?

d. What is the minimum distance achieved between the masses along their path?

Solution

Solution:
This is a two-body problem reduced to one-body with reduced mass $\mu = m/2$, total mass $M = 2m$, separation r, and angular momentum $L = \mu r^2 \dot{\theta}$.
Initial conditions: $r(0) = \ell$, $\dot{r}(0) = 0$, $L = m\ell v_0$ (each mass has $mv_0\ell/2$ about the CM).
Energy: $E = \frac{1}{2}\mu v_{\text{rel}}^2 - \frac{Gm^2}{r}$. Initially $v_{\text{rel}} = 2v_0$, so:

$$E = \frac{1}{2}\frac{m}{2}(2v_0)^2 - \frac{Gm^2}{\ell} = mv_0^2 - \frac{Gm^2}{\ell}$$

The effective potential is $V_{\text{eff}}(r) = \dfrac{L^2}{2\mu r^2} - \dfrac{Gm^2}{r}$, where $L = m\ell v_0$, $\mu = m/2$.
(a) The masses cannot collide for any $v_0 > 0$ because the centrifugal barrier $L^2/(2\mu r^2) \propto 1/r^2$ dominates the gravitational attraction $\propto -1/r$ as $r \to 0$, creating an infinite repulsive barrier. Only if $v_0 = 0$ (purely radial infall, $L = 0$) would a head-on collision occur.
(b) For circular orbits with constant separation $r = \ell$, the centripetal force equals gravity. In the reduced-mass frame:

$$\mu\frac{v_{\text{rel}}^2}{r} = \frac{Gm^2}{r^2} \quad \Rightarrow \quad \frac{m}{2}\frac{(2v_0)^2}{\ell} = \frac{Gm^2}{\ell^2} \quad \Rightarrow \quad v_0^2 = \frac{Gm}{2\ell}$$

(c) Closed (bound) orbits require $E < 0$:

$$mv_0^2 - \frac{Gm^2}{\ell} < 0 \quad \Rightarrow \quad v_0^2 < \frac{Gm}{\ell}$$

(d) The minimum distance is found from $E = V_{\text{eff}}(r_{\min})$ with $\dot{r} = 0$:

$$E = \frac{L^2}{2\mu r_{\min}^2} - \frac{Gm^2}{r_{\min}}$$

where $L = m\ell v_0$, $\mu = m/2$. This quadratic gives:

$$r_{\min} = \frac{Gm^2}{2|E|}\left(\sqrt{1 + \frac{2EL^2}{\mu G^2 m^4}} - 1\right)$$

For $E = 0$ (marginally bound): $r_{\min} = L^2/(2\mu G m^2) = \ell^2 v_0^2/(Gm)$.

2.0.5 Problem 5 (Hohmann Orbit Transfer to Mars)

Problem

A spaceship starts out in a circular orbit around the Sun very near the Earth (R_E) and aims to transfer to a circular orbit around the Sun very close to Mars (R_M). It makes this transfer via an elliptical Hohmann transfer orbit (shown in bold).

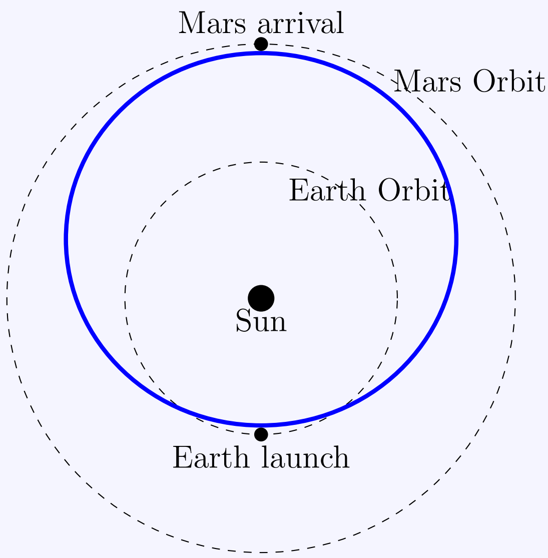

Figure 6: Hohmann Transfer Orbit from Earth to Mars.

This transfer is accomplished with an initial velocity boost Δv_1 near Earth and a second velocity boost Δv_2 near Mars. Assume both boosts are instantaneous impulses, and ignore the gravitational attraction of Earth and Mars themselves. Let $\alpha = R_M/R_E$.

i. Derive an expression for Δv_1 to change the orbit from circular to elliptical. Express in terms of v_E (Earth's orbital speed) and α.

ii. Derive an expression for Δv_2 to change the orbit from elliptical to circular at Mars. Express in terms of v_E and α.

iii. What is the angular separation between Earth and Mars, as measured from the Sun, at the time of launch so that the rocket will arrive exactly when Mars reaches that point? Express in terms of α.

Solution

Solution:
Earth's circular orbital speed: $v_E = \sqrt{GM_\odot/R_E}$. Mars' circular orbital speed: $v_M = \sqrt{GM_\odot/R_M} = v_E/\sqrt{\alpha}$.

The Hohmann transfer ellipse has perihelion R_E and aphelion R_M, so semi-major axis:

$$a = \frac{R_E + R_M}{2} = \frac{R_E(1+\alpha)}{2}$$

(i) At perihelion of the transfer ellipse, the spacecraft velocity is:

$$v_{p,\text{ell}} = \sqrt{GM_\odot\left(\frac{2}{R_E} - \frac{1}{a}\right)} = \sqrt{GM_\odot\left(\frac{2}{R_E} - \frac{2}{R_E(1+\alpha)}\right)} = v_E\sqrt{\frac{2\alpha}{1+\alpha}}$$

The boost needed from Earth's circular orbit:

$$\Delta v_1 = v_{p,\text{ell}} - v_E = v_E\left(\sqrt{\frac{2\alpha}{1+\alpha}} - 1\right)$$

(ii) At aphelion of the transfer ellipse:

$$v_{a,\text{ell}} = \sqrt{GM_\odot\left(\frac{2}{R_M} - \frac{1}{a}\right)} = \sqrt{GM_\odot\left(\frac{2}{R_M} - \frac{2}{R_E(1+\alpha)}\right)} = \frac{v_E}{\sqrt{\alpha}}\sqrt{\frac{2}{1+\alpha}}$$

Mars' circular speed is $v_M = v_E/\sqrt{\alpha}$. The boost needed to circularize:

$$\Delta v_2 = v_M - v_{a,\text{ell}} = \frac{v_E}{\sqrt{\alpha}}\left(1 - \sqrt{\frac{2}{1+\alpha}}\right)$$

(iii) The transfer time is half the period of the ellipse:

$$T_\text{trans} = \frac{1}{2}T_\text{ell} = \pi\sqrt{\frac{a^3}{GM_\odot}} = \pi\sqrt{\frac{R_E^3(1+\alpha)^3}{8GM_\odot}} = \frac{T_E}{4\sqrt{2}}(1+\alpha)^{3/2}$$

where $T_E = 2\pi\sqrt{R_E^3/(GM_\odot)}$ is Earth's orbital period.
During this time, Mars moves through an angle $\theta_M = 2\pi T_\text{trans}/T_M$, where $T_M = T_E\,\alpha^{3/2}$ is Mars' orbital period:

$$\theta_M = 2\pi\,\frac{T_\text{trans}}{T_E\alpha^{3/2}} = 2\pi\,\frac{(1+\alpha)^{3/2}}{4\sqrt{2}\,\alpha^{3/2}} = \frac{\pi}{2\sqrt{2}}\left(\frac{1+\alpha}{\alpha}\right)^{3/2}$$

The required angular separation between Earth and Mars at launch is $\pi - \theta_M$ (Mars must be ahead of Earth by this angle at launch):

$$\phi = \pi - \frac{\pi}{2\sqrt{2}}\left(\frac{1+\alpha}{\alpha}\right)^{3/2} = \pi\left[1 - \frac{1}{2\sqrt{2}}\left(\frac{1+\alpha}{\alpha}\right)^{3/2}\right]$$

For $\alpha = R_M/R_E = 1.524$, $\theta_M \approx 135.7^\circ$, so $\phi \approx 44.3^\circ$, meaning Mars must be about 44° ahead of Earth in its orbit at the time of launch.

2.0.6 Problem 6 (Relativistic Black Hole Potential)

Problem

In general relativity, the effective potential describing a particle of mass m orbiting a black hole of mass M is given by:

$$V(r) = -\frac{GMm}{r} - \frac{GML^2}{mc^2r^3}$$

The second term is a relativistic effect. (Note: The standard angular momentum barrier $L^2/(2mr^2)$ must still be added separately.)

(a) Explain why this new term allows particles to fall directly to the center ($r = 0$), and why this is impossible in Newtonian gravity.

(b) For fixed L, find the values of the circular orbit radii.

(c) Find the radius of the smallest possible stable circular orbit (ISCO). What happens if you try to orbit closer? (Answer: $6GM/c^2$)

(d) Find the closest possible approach radius of an unbound object. (Answer: $3GM/c^2$)

Solution

Solution:
The total effective potential including the centrifugal barrier is:

$$V_{\text{eff}}(r) = \frac{L^2}{2mr^2} - \frac{GMm}{r} - \frac{GML^2}{mc^2r^3}$$

(a) In Newtonian gravity, the centrifugal barrier $L^2/(2mr^2)$ dominates as $r \to 0$ ($\propto 1/r^2$ vs $-1/r$), creating an infinite repulsive barrier that prevents particles from reaching $r = 0$. The relativistic term is attractive and scales as $-1/r^3$, which dominates over the centrifugal $+1/r^2$ term for sufficiently small r. Hence, if a particle gets close enough, the net potential becomes attractive and pulls it into the singularity.
(b) Circular orbits satisfy $V'_{\text{eff}}(r) = 0$:

$$-\frac{L^2}{mr^3} + \frac{GMm}{r^2} + \frac{3GML^2}{mc^2r^4} = 0$$

Multiplying by r^4:

$$-\frac{L^2}{m}r + GMmr^2 + \frac{3GML^2}{mc^2} = 0$$

Multiplying by m: $-L^2r + GMm^2r^2 + 3GML^2/c^2 = 0$

$$GMm^2r^2 - L^2r + \frac{3GML^2}{c^2} = 0$$

Solving the quadratic:

$$r = \frac{L^2 \pm \sqrt{L^4 - 12G^2M^2m^2L^2/c^2}}{2GMm^2}$$

Two roots exist when $L \geq L_{\text{crit}} = 2\sqrt{3}\,GMm/c$.
(c) The ISCO occurs where the two roots coincide, i.e., the discriminant vanishes:

$$L^4 = \frac{12G^2M^2m^2L^2}{c^2} \quad \Rightarrow \quad L^2_{\text{ISCO}} = \frac{12G^2M^2m^2}{c^2}$$

Substituting back:

$$r_{\text{ISCO}} = \frac{L^2}{2GMm^2} = \frac{12G^2M^2m^2/c^2}{2GMm^2} = \frac{6GM}{c^2}$$

For $r < 6GM/c^2$, no stable circular orbit exists — any perturbation causes the particle to spiral into the black hole.
(d) For an unbound object, the closest possible approach is the innermost turning point of the effective potential. The two circular orbit radii from part (b) are:

$$r_{\pm} = \frac{L^2 \pm \sqrt{L^4 - 12G^2M^2m^2L^2/c^2}}{2GMm^2}$$

The smaller root r_- corresponds to an unstable circular orbit. In the limit $L \to \infty$:

$$r_- \approx \frac{L^2 - L^2(1 - 6G^2M^2m^2/(c^2L^2))}{2GMm^2} = \frac{6G^2M^2m^2/c^2}{2GMm^2} = \frac{3GM}{c^2}$$

This is the photon sphere — the innermost circular orbit for any particle. No object can have a stable turning point inside this radius; any approach closer than $3GM/c^2$ results in inevitable capture by the black hole.

3 Important Equations

Parameter	**Description**	**Formula / Definition**
Central force	General form	$\vec{F}(r) = F(r)\,\hat{r}$
Velocity (polar)	Radial & tangential	$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$
Acceleration (polar)	Radial & tangential	$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$
Angular momentum	Conserved quantity	$\vec{L} = m\vec{r} \times \vec{v}$, $l = mr^2\dot{\theta}$
Radial EOM	Newton's 2nd law	$m(\ddot{r} - r\dot{\theta}^2) = F(r)$
Total energy	Mechanical energy	$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r)$
Effective potential	1D radial reduction	$V_{\text{eff}}(r) = \dfrac{l^2}{2mr^2} + V(r)$
Binet formula	Orbit ODE	$\dfrac{d^2u}{d\theta^2} + u = -\dfrac{m}{l^2}F\left(\dfrac{1}{u}\right)\dfrac{1}{u^2}$
Inverse-square orbit	Conic section	$r(\theta) = \dfrac{l^2/(mk)}{1 + e\cos(\theta - \theta_0)}$
Eccentricity	Shape parameter	$e = \sqrt{1 + \dfrac{2El^2}{mk^2}}$
Radial oscillation	Stability frequency	$\Omega_r^2 = \dfrac{1}{m}\left.\dfrac{d^2V_{\text{eff}}}{dr^2}\right\|_{r=r_0}$

Key Insights

- Central forces are fundamental in physics, characterized by direction to or from a fixed center and dependence only on radius.
- Motion under central forces is restricted to a plane with **conserved angular momentum** and total mechanical energy.
- Forces like gravity and electrostatics follow the inverse-square law, yielding **orbital paths as conic sections** with shapes determined by total energy and angular momentum.
- The **effective potential** simplifies radial motion to one-dimensional analysis, highlighting stable and unstable configurations.
- Equations of motion express radial and angular dynamics, with solutions providing explicit parametric forms of the orbit.
- Small perturbations produce **radial oscillations** whose frequency informs orbit stability.
- These principles underpin classical planetary motion, satellite orbits, and atomic-scale force interactions.

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