Gravitation & Central Force Motion
Class 1 :3

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Key Topics and Concepts

1. Satellite Motion with Resistive Force Satellite motion under a resistive force opposing velocity is analyzed. The resistive force is modeled as proportional to the square of velocity:

$$F_{\text{res}} = \alpha v^2$$

Velocity v and radius r change with time, and the trajectory tends to spiral inward due to energy dissipation. The energy equation incorporates gravitational attraction and resistive dissipation:

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

Derived time expressions suggest how the radius and velocity evolve as resistive force causes orbital decay.

2. Pressure Inside a Uniform Solid Sphere (Gravitational Compression) A uniform solid sphere of mass m and radius R is considered to find pressure $P(r)$ as a function of radius r due to gravitational compression. Hydrostatic equilibrium gives the pressure gradient balancing gravitational force:

$$\frac{dP}{dr} = -\rho(r)\frac{GM(r)}{r^2}$$

For uniform density ρ, pressure inside the sphere varies quadratically:

$$P(r) = \frac{3GM^2}{8\pi R^4}\left(1 - \frac{r^2}{R^2}\right)$$

Maximum pressure occurs at the center $r = 0$ and decreases toward the surface.

3. Two-Body Problem and Center of Mass Dynamics Differential equations govern motions of masses M_1 and M_2 under mutual gravity:

$$M_1\ddot{\mathbf{R}}_1 = -G\frac{M_1M_2}{|\mathbf{R}_1 - \mathbf{R}_2|^3}(\mathbf{R}_1 - \mathbf{R}_2)$$

The center of mass $\mathbf{R}_C$ satisfies:

$$\mathbf{R}_C = \frac{M_1\mathbf{R}_1 + M_2\mathbf{R}_2}{M_1 + M_2}$$

Relative motion reduces to a central force problem with solutions describing elliptical, parabolic, or hyperbolic trajectories.

4. Orbital Trajectories and Energy Classifications Orbit types depend on total energy E:

- $E < 0$: Elliptical (bound) orbit
- $E = 0$: Parabolic (critical) orbit
- $E > 0$: Hyperbolic (unbound) orbit

Velocity and radius relations connect energy levels with trajectory shape through:

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

5. Gravitational Slingshot / Hyperbolic Orbits Hyperbolic trajectories describe gravitational slingshot maneuvers where spacecraft gain velocity by close planetary flybys. Conic section parameters:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Eccentricity e satisfies:

$$e = \sqrt{1 + \frac{2EL^2}{m(GMm)^2}} > 1$$

The velocity at infinity (v_∞) relates to orbital constants and influences energy gain via slingshot effect.

6. Cross Section for Planetary Capture Effective cross section A for a particle to be captured by a planet, considering gravitational focusing, is:

$$A = \pi R^2 \left(1 + \frac{2GM}{Rv^2}\right)$$

This accounts for both geometric and gravitational lensing effects increasing the capture radius beyond the physical planet radius.

1 Gravitation Class Notes

(IPhO Camp 2024)

1.1 Gravitation 01 - Session 19

1.1.1 Problem 1 (Resistive force acting on a satellite)

An Artificial Satellite of mass m of Moon's mass M revolves in a circular orbit whose radius exceeds the radius R of the moon n times. In this motion, satellite experiences a straight resistive force due to cosmic dust and force actually velocity dependent which is $F = \alpha v^2$, where α is a constant. Find how long satellite will stay in orbit until it falls onto the moon's surface.

Solution:
At every moment, the satellite will approximately be in a circular orbit of radius $r(t)$. To approach the problem, we note that the work done on the satellite by the resistive force would be equal to the orbital energy change of the satellite.

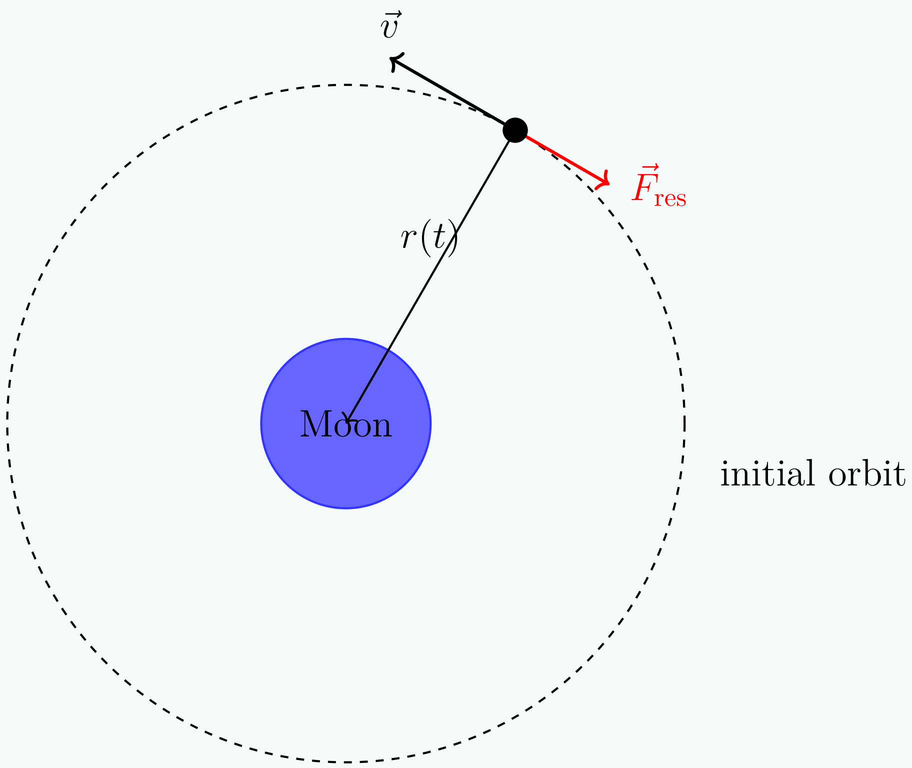

Figure 1: Satellite in circular orbit around Moon under resistive force. The orbit spirals inward as energy is dissipated.

$$
\begin{aligned}
dE = dW &= -\vec{F}\cdot d\vec{s} \\
\frac{GMm}{2r^2}dr &= -\alpha v^3 dt = -\alpha\left(\frac{GM}{r}\right)^{\frac{3}{2}} dt \\
\int_0^T dt &= -\frac{m}{2\alpha\sqrt{GM}}\int_{nR}^{R}\frac{1}{\sqrt{r}}\,dr \\
\therefore T &= \frac{m}{\alpha}\sqrt{\frac{R}{GM}}\left(\sqrt{n}-1\right) \\
&= \frac{m}{\alpha}\frac{\sqrt{n}-1}{\sqrt{gR}}
\end{aligned}
$$

Another approach is using Newton's second law on the external force directly and integrate from the initial circular orbit velocity to the final circular orbit velocity. Even though orbital velocity increases with decreasing radius whereas in our case velocity is decreasing instead, the result is identical due to symmetry.

$$
\begin{aligned}
a &= \frac{F_{\text{ext}}}{m} \\
\frac{dv}{dt} &= \frac{\alpha v^2}{m} \\
\int_{v_{nR}}^{v_R}\frac{1}{v^2}dv &= \int_0^T \frac{\alpha}{m}dt \\
\therefore T &= \frac{m}{\alpha}\left(\frac{1}{v_{nR}}-\frac{1}{v_R}\right) \\
&= \frac{m}{\alpha}\left(\sqrt{\frac{nR}{GM}}-\sqrt{\frac{R}{GM}}\right) \\
&= \frac{m}{\alpha}\frac{\sqrt{n}-1}{\sqrt{gR}}
\end{aligned}
$$

1.1.2 Problem 2 (Gravitational Compression)

A uniform solid sphere has a mass M and radius R.

(a) Find the pressure P inside the sphere, caused by gravitational compression as a function of distance r from its center.

(b) Find P at the center of the earth, assuming earth to be uniform solid sphere.

(c) Draw a graph of $P(r)$ vs r.

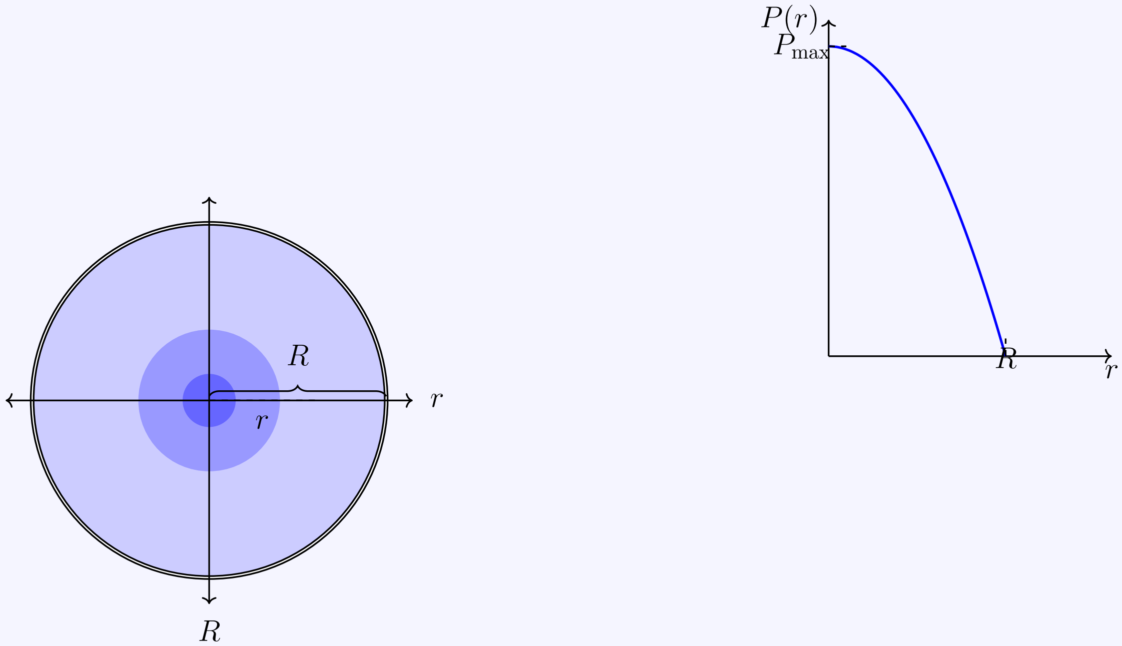

Figure 2: Left: Cross-sectional view of uniform solid sphere showing pressure increasing toward the center. Right: Quadratic pressure profile $P(r) = P_0(1-r^2/R^2)$.

Taking an infinitesimal shell element,

$$\frac{dp}{dr} = \frac{p(r+dr) - p(r)}{dr}$$

If the infinitesimal element has mass dm, we use an analysis using Newton's 2nd law.

$$\begin{aligned} dm\frac{d^2r}{dt^2} &= F(r+dr) - F(r) \\ &= F_g + F_p(r+dr) - F_p(r) \\ &= F_g + (p+dp)A - pA \\ &= F_g + A\,dp \end{aligned}$$

In equilibrium, the net force would be 0, giving $dm\frac{d^2r}{dt^2} = 0$. Therefore, using $F_g = -\frac{GM\,dm}{R^3}r$ and $dm = \rho\,dV = \rho A\,dr$, we get,

$$\begin{aligned} A\,dp &= -F_g = -\frac{GM\rho}{R^3}Ar\,dr \\ \therefore \frac{dp}{dr} &= -\frac{GM\rho}{R^3}r \end{aligned}$$

Now integrate from r to the surface R, where $P(R) = 0$:

$$\int_{P(r)}^{0} dp = -\frac{GM\rho}{R^3} \int_r^R r'\, dr'$$
$$-P(r) = -\frac{GM\rho}{R^3} \cdot \frac{R^2 - r^2}{2}$$
$$\therefore P(r) = \frac{GM\rho}{2R^3}(R^2 - r^2)$$

Using $\rho = \frac{M}{\frac{4}{3}\pi R^3} = \frac{3M}{4\pi R^3}$,

$$P(r) = \frac{GM}{2R^3} \cdot \frac{3M}{4\pi R^3}(R^2 - r^2)$$
$$= \frac{3GM^2}{8\pi R^6}(R^2 - r^2)$$
$$\therefore P(r) = \frac{3GM^2}{8\pi R^4}\left(1 - \frac{r^2}{R^2}\right)$$

The pressure at the center ($r = 0$) is

$$P(0) = \frac{3GM^2}{8\pi R^4}$$

1.1.3 Problem 3 (Binary System)

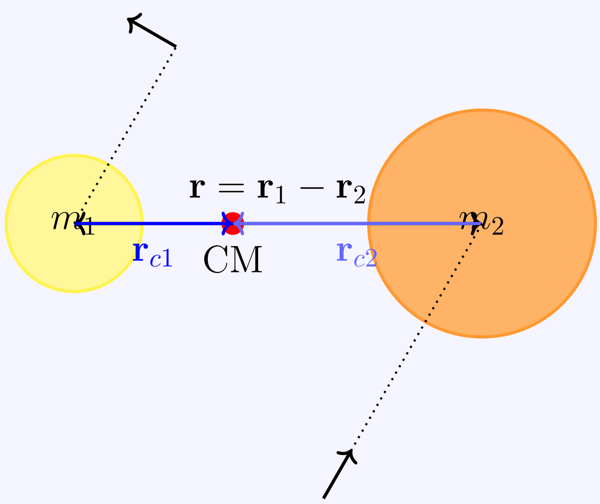

Figure 3: Binary star system with masses m_1 and m_2 orbiting their common center of mass. The relative position vector $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ and center-of-mass vectors are shown.

(a) Using Newton's 2nd law in vector form for both star 1 and 2,

$$
\begin{aligned}
m_1\ddot{\mathbf{r}}_1 &= \frac{Gm_1m_2}{|\mathbf{r}_1-\mathbf{r}_2|^3}(\mathbf{r}_2-\mathbf{r}_1)\\
\therefore \ddot{\mathbf{r}}_1 &= \frac{Gm_2}{|\mathbf{r}_1-\mathbf{r}_2|^3}(\mathbf{r}_2-\mathbf{r}_1)
\end{aligned}
$$

Similarly for m_2,

$$
\ddot{\mathbf{r}}_2 = \frac{Gm_1}{|\mathbf{r}_1-\mathbf{r}_2|^3}(\mathbf{r}_1-\mathbf{r}_2)
$$

(b) First we work with $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2$, using the expressions we found in (a),

$$
\begin{aligned}
\mathbf{r} &= \mathbf{r}_1-\mathbf{r}_2\\
\ddot{\mathbf{r}} &= \ddot{\mathbf{r}}_1-\ddot{\mathbf{r}}_2\\
&= \frac{Gm_2}{|\mathbf{r}_1-\mathbf{r}_2|^3}(\mathbf{r}_2-\mathbf{r}_1) - \frac{Gm_1}{|\mathbf{r}_1-\mathbf{r}_2|^3}(\mathbf{r}_1-\mathbf{r}_2)\\
&= -\frac{Gm_2}{|\mathbf{r}|^3}\mathbf{r} - \frac{Gm_1}{|\mathbf{r}|^3}\mathbf{r}\\
&= -\frac{G(m_1+m_2)}{|\mathbf{r}|^3}\mathbf{r}\\
\therefore \ddot{\mathbf{r}} + \frac{G(m_1+m_2)}{|\mathbf{r}|^3}\mathbf{r} &= 0
\end{aligned}
$$

Now, given equation for $\mathbf{r}_c(t)$,

$$
\mathbf{r}_c = \frac{m_1\mathbf{r}_1+m_2\mathbf{r}_2}{m_1+m_2}
$$

Taking the second derivative with respect to time on both sides gives,

$$
\begin{aligned}
(m_1+m_2)\ddot{\mathbf{r}}_c &= m_1\ddot{\mathbf{r}}_1+m_2\ddot{\mathbf{r}}_2\\
&= -m_1\frac{Gm_2}{|\mathbf{r}|^3}\mathbf{r} + m_2\frac{Gm_1}{|\mathbf{r}|^3}\mathbf{r} = 0\\
\therefore \ddot{\mathbf{r}}_c &= 0
\end{aligned}
$$

(c) From (b), $\ddot{\mathbf{r}}_c = 0$. And given the initial conditions $\dot{\mathbf{r}}_c(t=0)=0$ and $\mathbf{r}_c(t=0)=0$, the vector $\mathbf{r}_c$ must have a constant value of 0 throughout.

$$
\therefore \mathbf{r}_c(t) = 0
$$

(d) The differential equation for $\mathbf{r}$ as derived in (b), with $|\mathbf{r}|$ substituted with a:

$$
\ddot{\mathbf{r}} + \frac{G(m_1+m_2)}{a^3}\mathbf{r} = 0
$$

This second order vector differential equation can be written in the standard form $\ddot{\mathbf{r}}+\omega^2\mathbf{r}=0$, where

$$
\omega = \sqrt{\frac{G(m_1+m_2)}{a^3}}
$$

The solution of this equation has the form $\mathbf{r} = C_1 \cos(\omega t)\,\hat{x}_0 + C_2 \sin(\omega t)\,\hat{y}$. Using the initial condition $\mathbf{r}(t=0) = a\hat{x}_0$,

$$\mathbf{r} = a\cos(\omega t)\,\hat{x}_0 + a\sin(\omega t)\,\hat{y}$$

(e) If the distance between the stars is a and their orbital period is T, from Kepler's 3rd law,

$$T^2 = \frac{4\pi^2 a^3}{G(m_1+m_2)}$$
$$\therefore a = \left(\frac{G(m_1+m_2)T^2}{4\pi^2}\right)^{\frac{1}{3}}$$

1.1.4 Problem 4 (Capture cross section of a planet)

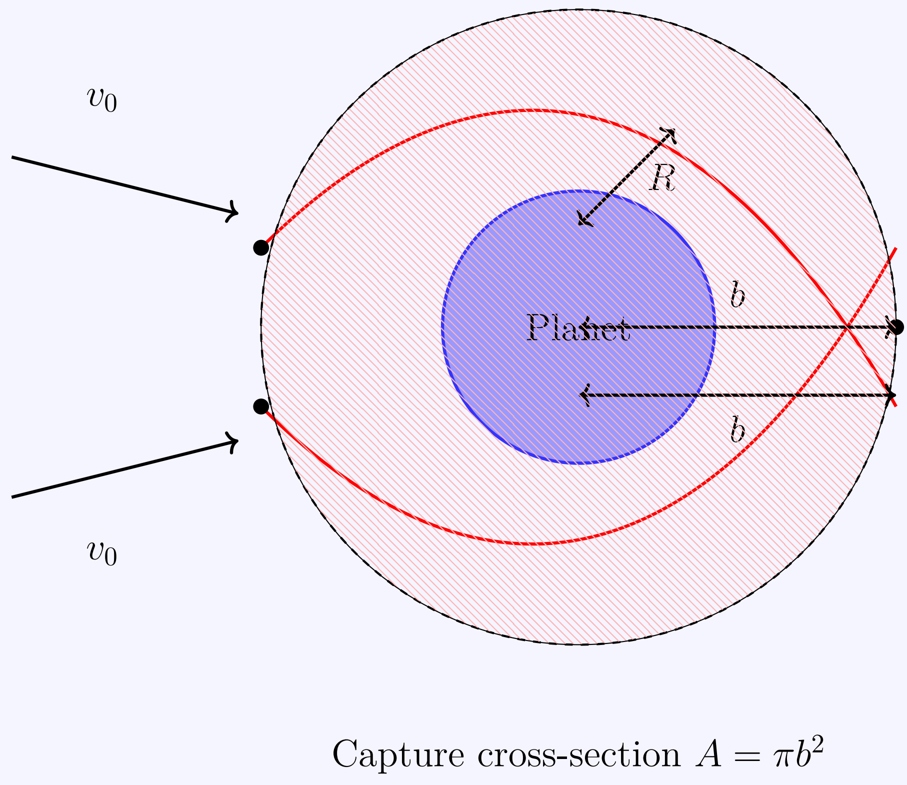

Figure 4: Geometry of gravitational capture cross section. Incoming particles with impact parameter b just graze the planet's surface R. Gravitational focusing increases the effective cross section beyond πR^2.

The capture cross section of a celestial body is defined as the area through which objects passing with a certain velocity will be on a trajectory to hit the body. The critical condition is a trajectory which just grazes the planet. Let us call the radius of this cross section b. The spacecraft has mass m, initial velocity v_0, velocity while grazing the planet v.

The only force on the spacecraft is the planet's gravitational force directed towards the center. Therefore, about the center of the planet, the net torque is 0 and angular momentum is conserved,

$$\begin{aligned} L_i &= L_f \\ m v_0 b &= m v R \\ v &= \frac{v_0 b}{R} \end{aligned} \tag{1}$$

There are no external force, therefore energy is conserved. We assume that the initial state is significantly far from the planet and thus the initial potential energy is negligible, $U_i \approx 0$.

$$\begin{aligned} K_i + U_i &= K_f + U_f \\ \frac{1}{2} m v_0^2 &= \frac{1}{2} m v^2 - \frac{GMm}{R} \end{aligned} \tag{2}$$

Substituting the expression for v from (1) into (2),

$$\begin{aligned} \frac{1}{2} m v_0^2 &= \frac{1}{2} m \left(\frac{v_0 b}{R} \right)^2 - \frac{GMm}{R} \\ \frac{v_0^2 b^2}{R^2} &= v_0^2 + \frac{2GM}{R} \\ \therefore b^2 &= R^2 \left(1 + \frac{2GM}{R v_0^2} \right) \end{aligned}$$

Therefore, capture cross section area,

$$A_b = \pi b^2 = \pi R^2 \left(1 + \frac{2GM}{R v_0^2} \right)$$

1.1.5 Problem 5 (Elliptical Projectile)

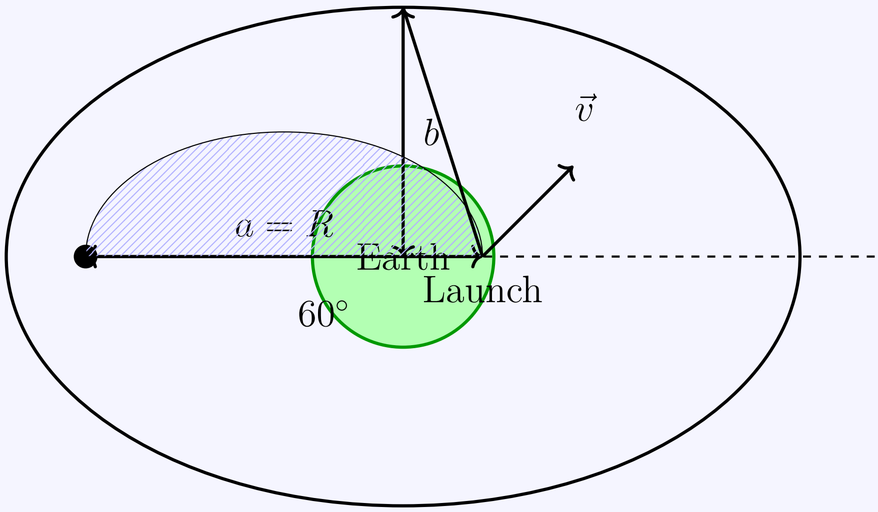

Figure 5: Elliptical projectile trajectory. The Earth's center is at one focus. The shaded region shows the area swept during the elliptical orbit. Launch angle $60°$ produces semi-minor axis $b = R\sin 60°$.

The trajectory of the rocket would be an elliptical path with one focus at the center of the Earth. For the ellipse,

$$\text{Semi-major axis of ellipse,} \quad a = R$$

$$\text{Semi-minor axis of ellipse,} \quad b = R\sin 60° = \frac{\sqrt{3}}{2}R$$

$$\therefore \text{Area of the ellipse } A_{\text{tot}} = \pi ab = \frac{\sqrt{3}}{2}\pi R^2$$

From the statement of Kepler's 2nd law we know that in a closed orbit, $\Delta t \propto \Delta A$, where ΔA is the area swept by the line from the focus of the ellipse in time Δt. Therefore,

$$\frac{T}{T_0} = \frac{A_{\text{ellipse}}}{A_{\text{swept}}} \quad \Rightarrow \quad T = \frac{A_{\text{swept}}}{A_{\text{tot}}}T_0$$

if the area of the shaded region is A_{swept}.

$$\begin{aligned} A_{\text{swept}} &= A_{\triangle OAB} + \frac{1}{2}A_{\text{ellipse}} \\ &= \frac{1}{2} \cdot 2b \cdot R\cos 60^\circ + \frac{1}{2} \cdot \frac{\sqrt{3}}{2}\pi R^2 \\ &= \frac{\sqrt{3}}{4}R^2 + \frac{\sqrt{3}}{4}\pi R^2 \\ &= \frac{\sqrt{3}R^2}{4}(1+\pi) \end{aligned}$$

$$\therefore T = T_0\left(\frac{1+\pi}{2\pi}\right) = \frac{(1+\pi)R^{\frac{3}{2}}}{\sqrt{GM}}$$

1.1.6 Problem 6 (Cloud of dust particles)

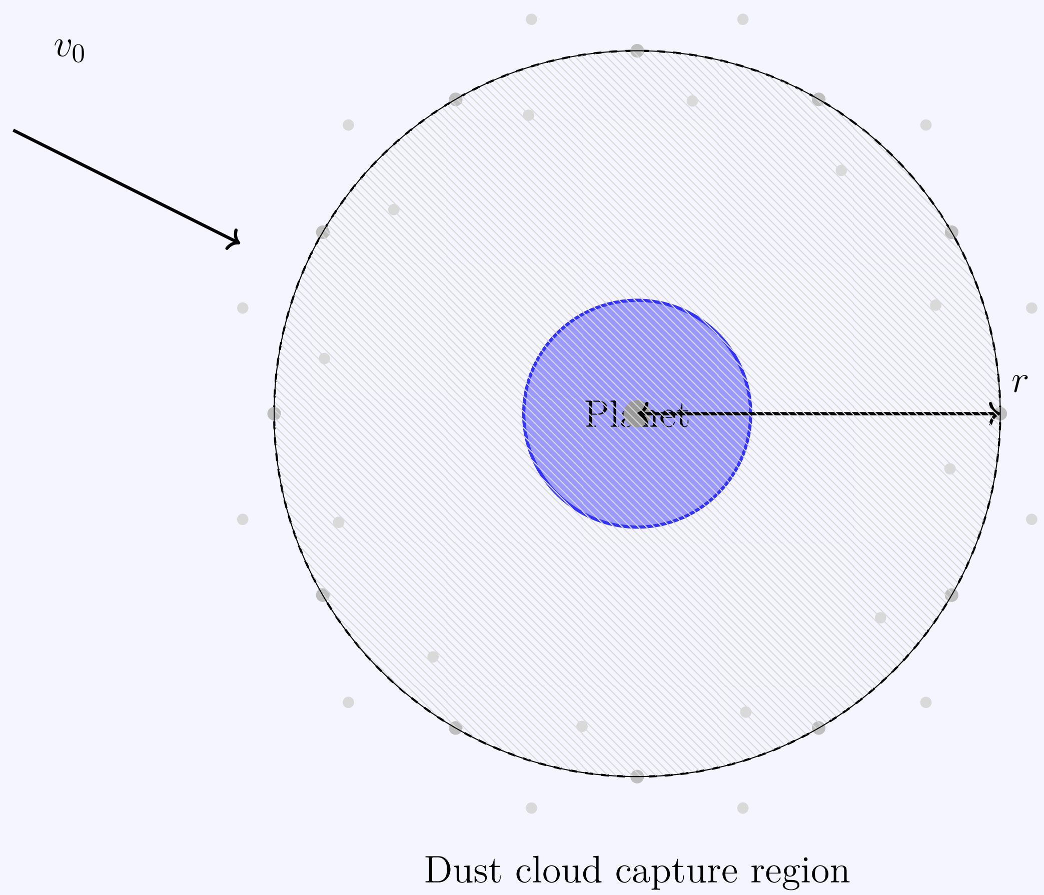

Figure 6: A planet moving through a dust cloud collects dust particles within its gravitational capture cross section. The effective capture radius r is determined by equating escape velocity with dust cloud velocity.

To find the amount of dust collected by the planet we need to find the radius of the maximum orbital capture cross section of the planet.

Escape velocity from a planet at a distance r from the center,

$$v_e(r) = \sqrt{\frac{2GM}{r}} = v_e\sqrt{\frac{R}{r}}, \qquad v_e = \sqrt{\frac{2GM}{R}}$$

We set this equal to the dust cloud velocity to get an expression for r,

$$v_0 = v_e\sqrt{\frac{R}{r}} \quad \Rightarrow \quad r = R\frac{v_e^2}{v_0^2}$$

In Problem 4, we derived the capture cross section for a planet of radius R as $b^2 = R^2\left(1 + \frac{2GM}{Rv_0^2}\right)$. Replacing the planet's radius R with the effective capture radius r, and using $v_0^2 = \dfrac{Rv_e^2}{r}$ from the escape condition, we get the square of the orbital capture cross section to be,

$$\begin{aligned} b^2 &= r^2\left(1 + \frac{2GM}{rv_0^2}\right) = r^2\left(1 + \frac{2GM}{Rv_e^2}\right) \\ &= \frac{R^2 v_e^4}{v_0^4}\left(1 + \frac{2GM}{Rv_e^2}\right) \end{aligned}$$

Finally, the mass of the collected dust is $m = V\rho = \pi b^2 l\rho$.

$$\therefore m_{\text{dust}} = \frac{\pi l R^2 \rho v_e^4}{v_0^4}\left(1 + \frac{2GM}{Rv_e^2}\right)$$

2 Central Force Motion - Session 23

Dynamics of Unbound Orbits

Parabolic Orbit ($E = 0$)

A parabolic orbit is the boundary between bound (elliptical) and unbound (hyperbolic) orbits. The total mechanical energy is zero:

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r} = 0$$

This gives the escape velocity condition at any distance r:

$$v_{\text{esc}}(r) = \sqrt{\frac{2GM}{r}}$$

The orbit equation for a parabola in polar coordinates (r, θ) takes the form:

$$r = \frac{2q}{1 + \cos\theta}$$

where q is the periapsis distance (closest approach). The eccentricity $e = 1$ exactly.

Energy proof for parabolic orbit. From the energy equation $E = \frac{1}{2}mv^2 - \frac{GMm}{r} = 0$, we have $v^2 = 2GM/r$. Using angular momentum conservation $L = mr^2\dot{\theta} =$ constant and the radial equation of motion, one can derive the polar equation above. Alternatively, from the vis-viva equation:

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)$$

Setting $a \to \infty$ (parabolic limit) gives $v^2 = 2GM/r$, consistent with $E = 0$. □

Hyperbolic Orbit ($E > 0$)

A hyperbolic orbit corresponds to $E > 0$, where the body has sufficient kinetic energy to escape the gravitational potential. The eccentricity satisfies $e > 1$.

Total energy proof for hyperbolic orbit. The total energy for an inverse-square central force is:

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

For a hyperbolic trajectory, $E > 0$. The orbit equation in polar form is:

$$r = \frac{a(e^2 - 1)}{1 + e\cos\theta}$$

where $a > 0$ is the semi-transverse axis and $e > 1$ is the eccentricity. The eccentricity relates to energy and angular momentum via:

$$e = \sqrt{1 + \frac{2EL^2}{m(GMm)^2}}$$

At infinity ($r \to \infty$), the potential vanishes and the kinetic energy gives the hyperbolic excess velocity v_∞:

$$\frac{1}{2} m v_\infty^2 = E \quad \Rightarrow \quad v_\infty = \sqrt{\frac{2E}{m}}$$

Thus $v^2 = v_\infty^2 + \dfrac{2GM}{r}$ at any point along the hyperbolic trajectory. $\square$

The turning angle δ (angle through which the velocity vector is deflected) in a hyperbolic flyby is:

$$\delta = 2 \arcsin\left(\frac{1}{e}\right) \quad \Rightarrow \quad \sin\frac{\delta}{2} = \frac{1}{e}$$

This result is essential for gravitational slingshot calculations (see Problem 7b).

2.0.1 Problem 7 (Hohmann with a slingshot)

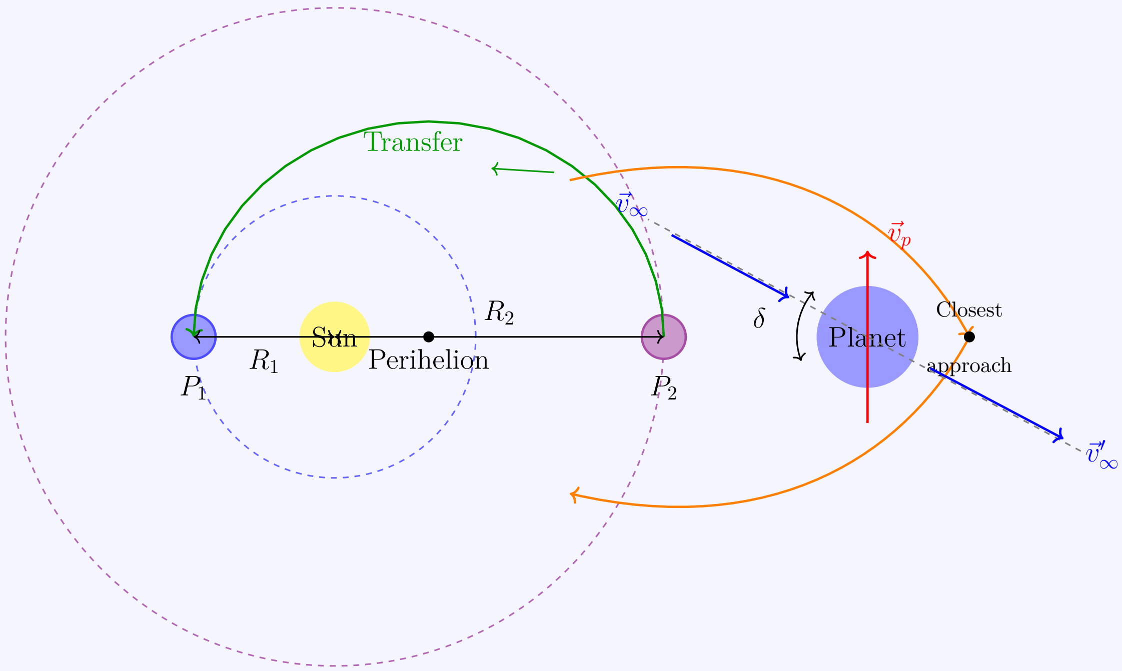

Figure 7: Hohmann transfer orbit (left) and gravitational slingshot (right). The transfer ellipse connects P_1 at R_1 to P_2 at R_2. The slingshot deflects the spacecraft's relative velocity by δ, changing its heliocentric velocity from $\vec{v}_2$ to $\vec{v}_2'$.

(a) The minimal energy transfer from Planet 1 to Planet 2 would be an elliptical

orbit with perihelion distance R_1 and aphelion distance R_2.
At the perihelion point on an elliptical orbit, the velocity vector is perpendicular to the orbit. Launching the probe parallel to the Earth's velocity in the solar frame gives the correct velocity direction for the low-energy transfer orbit, and thus minimizes the energy needed.

(b) Gravitational slingshot at Planet 2.
The spacecraft arrives at aphelion of the Hohmann transfer ellipse ($r = R_2$). Its velocity there is found from the vis-viva equation with $a = (R_1 + R_2)/2$:

$$v_2^2 = GM_\odot\left(\frac{2}{R_2} - \frac{2}{R_1 + R_2}\right) \quad \Rightarrow \quad v_2 = \sqrt{\frac{2GM_\odot R_1}{R_2(R_1 + R_2)}}$$

Planet 2 orbits the Sun on a circular orbit of radius R_2 with velocity:

$$V_2 = \sqrt{\frac{GM_\odot}{R_2}}$$

Since $R_2 > R_1$, we have $v_2 < V_2$: the spacecraft is slower than the planet at encounter.

Relative velocity (planet frame): The spacecraft approaches the planet from ahead (i.e., the planet catches up from behind). In the planet's rest frame:

$$\mathbf{u} = \mathbf{v}_2 - \mathbf{V}_2 \quad \Rightarrow \quad u = |\mathbf{u}| = V_2 - v_2$$

Hyperbolic flyby: In the planet's frame, the trajectory is a hyperbola about Planet 2 (mass M_p). The speed is conserved:

$$|\mathbf{u}'| = |\mathbf{u}| = u$$

Let the closest approach distance be d (typically a few times the planet's radius). The eccentricity of the flyby hyperbola is:

$$e = 1 + \frac{d\,u^2}{GM_p}$$

The turning angle δ (angle between $\mathbf{u}$ and $\mathbf{u}'$) is:

$$\delta = 2\arcsin\left(\frac{1}{e}\right) \quad \Rightarrow \quad \sin\frac{\delta}{2} = \frac{1}{e}$$

Outgoing velocity (Sun frame): After the flyby, the relative velocity vector is rotated by δ. Choosing coordinates with x along $\mathbf{V}_2$, the incoming relative velocity is $\mathbf{u} = (-u, 0)$ (spacecraft slower than planet). After a trailing-side flyby, the outgoing relative velocity is:

$$\mathbf{u}' = u\,(\cos(\pi-\delta),\ \sin(\pi-\delta)) = (-u\cos\delta,\ u\sin\delta)$$

Transforming back to the Sun's frame:

$$\mathbf{v}_2' = \mathbf{V}_2 + \mathbf{u}' = (V_2 - u\cos\delta,\ u\sin\delta)$$

The magnitude of the outgoing heliocentric velocity is:

$$v_2'^2 = V_2^2 + u^2 - 2V_2 u\cos\delta$$

Energy gain: The spacecraft's specific energy change due to the slingshot is:

$$\Delta E = \frac{1}{2}m(v_2'^2 - v_2^2)$$

For a near-180° deflection ($\delta \approx \pi$, i.e., $e \lesssim 1$), we have $\cos\delta \approx -1$ and:

$$v_2'^2 \approx V_2^2 + u^2 + 2V_2 u = (V_2+u)^2 \quad \Rightarrow \quad v_2' \approx V_2 + u = 2V_2 - v_2$$

The energy gain in this optimal case is:

$$\Delta E_{\max} = \frac{1}{2}m\big((2V_2 - v_2)^2 - v_2^2\big) = 2mV_2(V_2 - v_2) = 2mV_2 u$$

This slingshot accelerates the spacecraft to a higher-energy heliocentric orbit, potentially enabling travel to outer planets or escape from the Solar System.

3 Two Body Problem 01 - Session 26

3.1 Reduction of Two-Body Problem

Consider two masses m_1 and m_2 interacting via mutual gravitational attraction. The equations of motion in an inertial frame are:

$$m_1\ddot{\mathbf{r}}_1 = -\frac{Gm_1m_2}{|\mathbf{r}_1-\mathbf{r}_2|^3}(\mathbf{r}_1-\mathbf{r}_2), \qquad m_2\ddot{\mathbf{r}}_2 = -\frac{Gm_1m_2}{|\mathbf{r}_1-\mathbf{r}_2|^3}(\mathbf{r}_2-\mathbf{r}_1)$$

3.2 Center of Mass Coordinates

Define the center of mass position:

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1+m_2}$$

Adding the equations of motion gives $\ddot{\mathbf{R}} = 0$, i.e., the center of mass moves with constant velocity.

3.3 Relative Motion and Reduced Mass

Define the relative position vector $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$. The equation for relative motion is:

$$\ddot{\mathbf{r}} = \ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2 = -\frac{G(m_1 + m_2)}{r^3}\,\mathbf{r}$$

Multiplying both sides by the reduced mass $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$ gives:

$$\mu\,\ddot{\mathbf{r}} = -\frac{G m_1 m_2}{r^3}\,\mathbf{r} = -\frac{GM\mu}{r^3}\,\mathbf{r}$$

where $M = m_1 + m_2$ is the total mass. This is the equation of a single particle of mass μ moving in a fixed central potential $U(r) = -GM\mu/r$.

3.4 Constants of Motion

For the reduced one-body problem, two quantities are conserved:

Angular momentum:

$$\mathbf{L} = \mu\,\mathbf{r} \times \dot{\mathbf{r}} = \text{constant}$$

The magnitude $L = \mu r^2\dot{\theta}$ gives Kepler's second law: $\dfrac{dA}{dt} = \dfrac{L}{2\mu} = \text{constant}$.

Energy:

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r^2} - \frac{GM\mu}{r} = \text{constant}$$

3.5 Effective Potential

The radial motion can be analyzed using the effective potential:

$$U_{\text{eff}}(r) = \frac{L^2}{2\mu r^2} - \frac{GM\mu}{r}$$

The term $L^2/(2\mu r^2)$ is the centrifugal barrier. The total energy is:

$$E = \frac{1}{2}\mu\dot{r}^2 + U_{\text{eff}}(r)$$

Orbit types are classified by comparing E with the minimum of $U_{\text{eff}}(r)$:

- $E < U_{\text{eff,min}}$: Not possible
- $E = U_{\text{eff,min}}$: Circular orbit ($r = \text{constant}$)
- $U_{\text{eff,min}} < E < 0$: Elliptical (bound) orbit
- $E = 0$: Parabolic orbit
- $E > 0$: Hyperbolic (unbound) orbit

The effective potential minimum occurs at $r_0 = L^2/(GM\mu^2)$ with value $U_{\text{eff,min}} = -G^2M^2\mu^3/(2L^2)$.

4 Two Body Problem 02 - Session 27

4.1 Orbit Equation (Binet Formula)

For a central force $F(r) = -GM\mu/r^2$, the orbit equation is obtained via the Binet formula:

$$\frac{d^2u}{d\theta^2} + u = -\frac{F(1/u)}{\mu L^2 u^2} = \frac{GM\mu}{L^2}$$

where $u = 1/r$ and $L = \mu r^2\dot{\theta}$ is the constant angular momentum. The solution is a conic section:

$$u = \frac{GM\mu^2}{L^2}\big(1 + e\cos(\theta - \theta_0)\big)$$

or equivalently:

$$r = \frac{L^2/(GM\mu^2)}{1 + e\cos(\theta - \theta_0)} = \frac{a(1-e^2)}{1 + e\cos(\theta - \theta_0)}$$

4.2 Classification of Orbits

Setting $\theta_0 = 0$ and defining the semi-latus rectum $p = L^2/(GM\mu^2) = a(1-e^2)$:

$$r(\theta) = \frac{p}{1 + e\cos\theta}$$

The orbit type is determined by eccentricity e:

- $e = 0$: Circular orbit ($r = p = a$)
- $0 < e < 1$: Elliptical orbit (bound, $E < 0$)
- $e = 1$: Parabolic orbit ($E = 0$)
- $e > 1$: Hyperbolic orbit (unbound, $E > 0$)

4.3 Kepler's Laws

First Law (Ellipse Law): Each planet moves in an ellipse with the Sun at one focus. Follows directly from the conic solution $r(\theta) = p/(1 + e\cos\theta)$.

Second Law (Area Law): The radius vector sweeps out equal areas in equal times. From angular momentum conservation:

$$\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{L}{2\mu} = \text{constant}$$

Third Law (Harmonic Law): The square of the orbital period is proportional to the cube of the semi-major axis:

$$T^2 = \frac{4\pi^2 a^3}{G(m_1 + m_2)}$$

Proof. The area of an ellipse is $A = \pi ab = \pi a^2\sqrt{1 - e^2}$. Using Kepler's second law:

$$T = \frac{A}{dA/dt} = \frac{\pi a^2\sqrt{1 - e^2}}{L/(2\mu)} = \frac{2\pi\mu a^2\sqrt{1 - e^2}}{L}$$

From the orbit equation, $L^2 = GM\mu^2 a(1 - e^2)$. Substituting:

$$T = \frac{2\pi a^{3/2}}{\sqrt{GM}} \quad \Rightarrow \quad T^2 = \frac{4\pi^2 a^3}{GM}$$

where $M = m_1 + m_2$ is the total mass. □

4.4 Vis-Viva Equation

A fundamental relation connecting velocity and position in any Keplerian orbit:

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)$$

where a is the semi-major axis (for hyperbolas, $a < 0$). This equation unifies all conic-section orbits and follows directly from energy conservation.

4.5 Orbital Elements

A Keplerian orbit is described by six orbital elements:

- a: Semi-major axis (orbit size)
- e: Eccentricity (orbit shape)
- i: Inclination (tilt relative to reference plane)
- Ω: Longitude of ascending node (orientation in reference plane)
- ω: Argument of periapsis (orientation of ellipse in orbital plane)
- T_0: Time of periapsis passage (timing)

4.6 Kepler's Equation

The mean anomaly $M = n(t - T_0)$, where $n = 2\pi/T$ is the mean motion, relates to the eccentric anomaly E via:

$$M = E - e \sin E$$

The true anomaly θ is then obtained from:

$$\tan \frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2}$$

This transcendental equation must be solved numerically for general e.

5 Important Equations

Parameter	**Description**	**Formula / Definition**
F_{res}	Resistive force	αv^2
E	Total orbital energy	$E = \frac{1}{2}mv^2 - \frac{GMm}{r}$
$P(r)$	Pressure inside uniform sphere	$P(r) = \frac{3GM^2}{8\pi R^4}\left(1 - \frac{r^2}{R^2}\right)$
$\mathbf{R}_C$	Center of mass	$\frac{M_1\mathbf{R}_1 + M_2\mathbf{R}_2}{M_1 + M_2}$
Orbital Energy	Trajectory type	Elliptical: $E < 0$, Parabolic: $E = 0$, Hyperbolic: $E > 0$
Cross Section	Capture cross section	$A = \pi R^2\left(1 + \frac{2GM}{Rv^2}\right)$
Hyperbola	Equation of hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
Eccentricity e	Hyperbolic eccentricity	$e = \sqrt{1 + \frac{2EL^2}{m(GMm)^2}}$

Key Insights

- Resistive forces in space decay orbital radius and velocity, transforming initially circular orbits into spiral trajectories.
- The pressure inside a uniform solid body from self-gravity peaks at the center and diminishes toward the surface, conforming to hydrostatic balancing.
- Two-body dynamics can be reduced via center of mass coordinates, simplifying equations of motion to effective one-body central force problems.
- The shape of an orbit is fully determined by total mechanical energy, with parabolic orbits representing the escape condition.
- Hyperbolic orbits underpin gravitational slingshot maneuvers, providing vital velocity boosts to spacecraft via planetary flybys.
- The effective capture cross section of planets increases due to gravitational focusing, exceeding the planet's physical cross section, critical for understanding orbital capture and meteorite impact probabilities.

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