Problem 8: Divisibility Powers

Problem 1. *Find all positive integers $n < 10^{100}$ for which simultaneously n divides 2^n, $n-1$ divides 2^{n-1}, and $n-2$ divides 2^{n-2}.*

Solution

We are given
$$n \mid 2^n, \qquad n-1 \mid 2^{n-1}, \qquad n-2 \mid 2^{n-2},$$
and must find all $n < 10^{100}$.

We use the standard fact:
$$2^m - 1 \mid 2^k - 1 \iff m \mid k.$$

Step 1

From $n \mid 2^n$, every prime divisor of n must be 2. Hence
$$n = 2^k, \qquad k \geq 0.$$

Step 2

Now
$$n - 1 = 2^k - 1.$$
The condition
$$n - 1 \mid 2^{n-1}$$
becomes
$$2^k - 1 \mid 2^{2^k - 1} - 1.$$
By the stated fact, this holds iff
$$k \mid 2^k.$$
Thus
$$k = 2^\ell, \qquad \ell \geq 0.$$

Step 3

Now
$$n = 2^{2^\ell}.$$
The final condition
$$n - 2 \mid 2^{n-2}$$
becomes
$$2^{2^\ell} - 2 \mid 2^{2^{2^\ell} - 2}.$$

Dividing both sides by 2,

$$2^{2^{\ell}-1} - 1 \mid 2^{2^{2^{\ell}-1}-1} - 1.$$

Applying the same fact again gives

$$2^{\ell} - 1 \mid 2^{2^{\ell}-1} - 1,$$

which implies

$$\ell \mid 2^{\ell}.$$

Hence

$$\ell = 2^m, \qquad m \geq 0.$$

Therefore

$$n = 2^{2^{2^m}}.$$

Step 4: Bounding by 10^{100}

Evaluate:

$$\begin{aligned}
m = 0: &\quad n = 2^2 = 4,\\
m = 1: &\quad n = 2^4 = 16,\\
m = 2: &\quad n = 2^{16},\\
m = 3: &\quad n = 2^{256}.
\end{aligned}$$

For $m = 4$,

$$n = 2^{65536} > 10^{100},$$

so no larger m is allowed.

Answer

$$\boxed{n = 4,\ 16,\ 2^{16},\ 2^{256}}.$$

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