Problem 7: The Limp Rook

Problem 1. *On a 999×999 board a limp rook can move in the following way: From any square it can move to any of its adjacent squares, i.e. a square having a common side with it, and every move must be a turn, i.e. the directions of any two consecutive moves must be perpendicular. A non-intersecting route of the limp rook consists of a sequence of pairwise different squares that the limp rook can visit in that order by an admissible sequence of moves. Such a non-intersecting route is called cyclic, if the limp rook can, after reaching the last square of the route, move directly to the first square of the route and start over.*

How many squares does the longest possible cyclic, non-intersecting route of a limp rook visit?

Answer

Answer: $4 \cdot (499^2 - 1) = 998^2 - 4$.

Solution:

The answer is $998^2 - 4 = 4 \cdot (499^2 - 1)$ squares.

First we show that this number is an upper bound for the number of cells a limp rook can visit. To do this we color the cells with four colors A, B, C and D in the following way: for $(i, j) \equiv (0, 0) \pmod 2$ use A, for $(i, j) \equiv (0, 1) \pmod 2$ use B, for $(i, j) \equiv (1, 0) \pmod 2$ use C and for $(i, j) \equiv (1, 1) \pmod 2$ use D.

From an A-cell the rook has to move to a B-cell or a C-cell. In the first case, the order of the colors of the cells visited is given by $A, B, D, C, A, B, D, C, A, \ldots$, in the second case it is $A, C, D, B, A, C, D, B, A, \ldots$. Since the route is closed it must contain the same number of cells of each color. There are only 499^2 A-cells. In the following we will show that the rook cannot visit all the A-cells on its route and hence the maximum possible number of cells in a route is $4 \cdot (499^2 - 1)$.

Proof that Not All A-cells Can Be Visited

Assume that the route passes through every single A-cell. Color the A-cells in black and white in a chessboard manner, i.e. color any two A-cells at distance 2 in different color. Since the number of A-cells is odd the rook cannot always alternate between visiting black and white A-cells along its route. Hence there are two A-cells of the same color which are four rook-steps apart that are visited directly one after the other. Let these two A-cells have row and column numbers (a, b) and $(a + 2, b + 2)$ respectively.

There is up to reflection only one way the rook can take from (a, b) to $(a + 2, b + 2)$. Let this way be

$$(a, b) \to (a, b + 1) \to (a + 1, b + 1) \to (a + 1, b + 2) \to (a + 2, b + 2).$$

Also let without loss of generality the color of the cell $(a, b + 1)$ be B (otherwise change the roles of columns and rows).

Now consider the A-cell $(a, b+2)$. The only way the rook can pass through it is via

$$(a-1, b+2) \to (a, b+2) \to (a, b+3)$$

in this order, since according to our assumption after every A-cell the rook passes through a B-cell. Hence, to connect these two parts of the path, there must be a path connecting the cell $(a, b+3)$ and (a, b) and also a path connecting $(a+2, b+2)$ and $(a-1, b+2)$.

Contradiction by Intersection

But these four cells are opposite vertices of a convex quadrilateral and the paths are outside of that quadrilateral and hence they must intersect. This is due to the following fact:

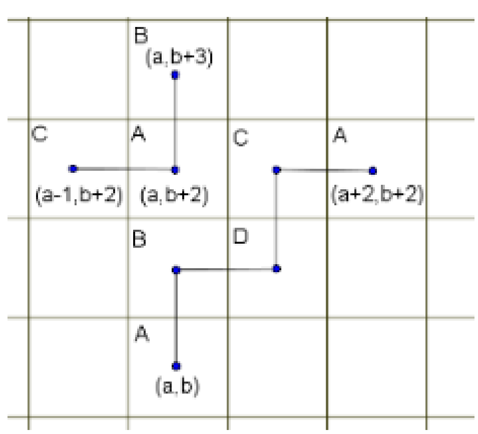
The path from (a, b) to $(a, b+3)$ together with the line segment joining these two cells form a closed loop that has one of the cells $(a-1, b+2)$ and $(a+2, b+2)$ in its inside and the other one on the outside. Thus the path between these two points must cross the previous path.

But an intersection is only possible if a cell is visited twice. This is a contradiction.

Hence the number of cells visited is at most $4 \cdot (499^2 - 1)$.

Constructive Proof

The following picture indicates a recursive construction for all $n \times n$-chessboards with $n \equiv 3 \pmod 4$ which clearly yields a path that misses exactly one A-cell (marked with a dot, the center cell of the 15×15-chessboard) and hence, in the case of $n = 999$ crosses exactly $4 \cdot (499^2 - 1)$ cells.



Therefore, we conclude that the maximum number of squares visited is $\boxed{4 \cdot (499^2 - 1)}$.

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