Problem 5: A Trigonometric Equation

Problem 1. *Solve over* $\mathbb{R}$ *the equation*

$$4^{(\sin x)^2} + 3^{(\tan x)^2} = 4^{(\cos x)^2} + 3^{(\cot x)^2}.$$

Answer

Answer: $x = \frac{\pi}{4} + k\frac{\pi}{2}$ for $k \in \mathbb{Z}$.

Solution: Setting $t = \cos^2 x \in (0,1)$, we have:

$$\begin{aligned}
\sin^2 x &= 1 - t, \\
\tan^2 x &= \frac{\sin^2 x}{\cos^2 x} = \frac{1-t}{t}, \\
\cot^2 x &= \frac{\cos^2 x}{\sin^2 x} = \frac{t}{1-t}.
\end{aligned}$$

Substituting these into the equation, we obtain:

$$4^{1-t} + 3^{\frac{1-t}{t}} = 4^t + 3^{\frac{t}{1-t}}.$$

Let $f(t) = 4^t + 3^{\frac{t}{1-t}}$ and $g(t) = 4^{1-t} + 3^{\frac{1-t}{t}}$. We need to solve $g(t) = f(t)$.
Observe that:

- The function $f(t) = 4^t + 3^{\frac{t}{1-t}}$ is increasing on $(0,1)$, since both 4^t and $3^{\frac{t}{1-t}}$ are increasing.
- The function $g(t) = 4^{1-t} + 3^{\frac{1-t}{t}}$ is decreasing on $(0,1)$, since 4^{1-t} is decreasing and $3^{\frac{1-t}{t}}$ is decreasing.

Since $f(t)$ is increasing and $g(t)$ is decreasing, they can intersect at most once.
We can verify that $t = \frac{1}{2}$ is a solution:

$$g\left(\frac{1}{2}\right) = 4^{1/2} + 3^1 = 2 + 3 = 5, \quad f\left(\frac{1}{2}\right) = 4^{1/2} + 3^1 = 2 + 3 = 5.$$

Therefore, $t = \frac{1}{2}$ is the unique solution, which gives $\cos^2 x = \frac{1}{2}$.
This implies $\cos x = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}$, so

$$\boxed{x = \frac{\pi}{4} + k\frac{\pi}{2} \text{ for } k \in \mathbb{Z}}. \quad \blacksquare$$

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