Problem 4: Counting Special Permutations

Problem 1. *Find the number of permutations* $(p_1, p_2, p_3, p_4, p_5, p_6)$ *of* $1, 2, 3, 4, 5, 6$ *such that for any* k*,* $1 \leq k \leq 5$*,* $(p_1, \ldots, p_k)$ *does not form a permutation of* $1, 2, \ldots, k$*.*

Answer

Answer: 461.

Solution: Let's denote the set of permutations of $\{1, \ldots, n\}$ as P_n, where $|P_n| = n!$ is the number of permutations. For a permutation $P = (\alpha_1 \ldots \alpha_n)$, let's define the *degree of the permutation*, denoted as $\deg P$. (Note: The degree of a permutation can be a different concept in the literature; here we use our own definition.) For $1 \leq k \leq n$, if $(\alpha_1 \ldots \alpha_k)$ represents a permutation of the set $\{1, \ldots, k\}$, the degree is the smallest k for which this holds.

Let Q_n denote the set of permutations of $\{1, \ldots, n\}$ such that, for every $1 \leq k \leq n-1$, $(\alpha_1 \ldots \alpha_k)$ is not a permutation of $\{1, \ldots, k\}$. For example, $Q_1 = \{(1)\}$ and $Q_2 = \{(21)\}$. The problem asks for $|Q_6|$.

For $1 \leq i \leq n$, let $D_i = \{P \in P_n \mid \deg P = i\}$. Note that D_i and D_j are disjoint for $i \neq j$, and $P_n = D_1 \cup \cdots \cup D_n$. First, observe that $D_n = \{P \in P_n \mid \deg P = n\} = Q_n$.

If we append a permutation of $\{i+1, \ldots, n\}$ to Q_i, we obtain a permutation in D_i. Therefore, we have the equation:

$$n! = |P_n| = |Q_n| + |Q_{n-1}| \cdot 1! + |Q_{n-2}| \cdot 2! + \cdots + |Q_1| \cdot (n-1)! = \sum_{i=1}^{n} |Q_i| \cdot (n-i)!$$

Let's calculate for smaller values:

$$\begin{aligned} |P_3| = 3! &= |Q_3| + |Q_2| \cdot 1! + |Q_1| \cdot 2! \\ \implies |Q_3| &= 3! - 1 - 2 = 3. \end{aligned}$$

$$\begin{aligned} |P_4| = 4! &= |Q_4| + |Q_3| \cdot 1! + |Q_2| \cdot 2! + |Q_1| \cdot 3! \\ \implies |Q_4| &= 4! - 3 - 2 - 6 = 13. \end{aligned}$$

$$\begin{aligned} |P_5| = 5! &= |Q_5| + |Q_4| \cdot 1! + |Q_3| \cdot 2! + |Q_2| \cdot 3! + |Q_1| \cdot 4! \\ \implies |Q_5| &= 5! - 13 - 6 - 6 - 24 = 71. \end{aligned}$$

$$\begin{aligned} |P_6| = 6! &= |Q_6| + |Q_5| \cdot 1! + |Q_4| \cdot 2! + |Q_3| \cdot 3! + |Q_2| \cdot 4! + |Q_1| \cdot 5! \\ \implies |Q_6| &= 6! - 71 - 26 - 18 - 24 - 120 = 461. \quad \blacksquare \end{aligned}$$

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