

Problem 3: A Number Writing Game

Problem 1. *Onyx and Sehtaj play a game. Given an integer N , Onyx writes down 1 first, then every player sees the last number written and if it is n then in her turn she writes $n + 1$ or $2n$, but her number cannot be bigger than N . The player who writes N wins. For which values of N does Sehtaj win?*

Answer

Answer: Sehtaj wins if and only if $N = 2^k$ and k is odd.

Solution: If N is odd, Onyx wins. She first writes 1, an odd number. If she writes an odd number n , both $n + 1$ and $2n$ are even, so the number she gets is even and thus she can make the $+1$ move, writing another odd number. Thus Sehtaj only writes even numbers and never N .

Let $N = d_1 d_2 \cdots d_k$ in binary, and suppose N is even. Let $a_i = d_1 d_2 \cdots d_{k-i+1} + 1$. Partition the set $\{1, 2, \dots, N\}$ into $S_1 = \{N, N - 1, \dots, a_2\}$, $S_2 = \{a_2 - 1, \dots, a_3\}$, and so on. A move $n \rightarrow 2n$ always goes to the following subset. Now it is clear that if a player is the first to write a number in S_1 , doing it by doubling the previous number, she wins (because she writes the even numbers from then on). If X is the first to write a number in S_2 , Y will then jump to S_1 by an $n \rightarrow 2n$ move and win. So the first to write a number in S_2 loses. We have $S_2 = \{d_1 \cdots d_{k-1}, \dots, d_1 \cdots d_{k-2} + 1\}$. Clearly $d_1 \cdots d_{k-2}$ will be written, unless a player commits suicide, because the only way to skip it is to play an $n \rightarrow 2n$ move and that is losing. So for even N the player who writes $d_1 \cdots d_{k-2}$ wins.

Then the winner for the $d_1 \cdots d_k$ game is Onyx if $d_k = 1$, and it is the same as for the $d_1 \cdots d_{k-2}$ game if $d_k = 0$. If among d_k, d_{k-2}, d_{k-4} and so on there's a 1, the winner is Onyx. If not, and if k is even, the winner is the same as for $d_1 d_2$, which is Onyx if $d_2 = 1$, and Sehtaj if $d_2 = 0$, by inspection. If k is odd, the winner is either Onyx or the same as for $d_1 d_2 d_3$, which is always Onyx by inspection.

We conclude that Sehtaj wins if and only if N is the sum of distinct odd powers of 2. In particular, Sehtaj wins if and only if $N = 2^k$ and k is odd. ■

proposed by Tawhid Bin Omar