Problem 1 (Functional Equation)**.** *Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real numbers x and y,*

$$f(x + f(y)) = f(x) + y$$

[ফাংশনাল সমীকরণ] এমন সকল অবিচ্ছিন্ন (continuous) ফাংশন $f : \mathbb{R} \to \mathbb{R}$ নির্ণয় কর যেন সকল বাস্তব সংখ্যা x এবং y এর জন্য,

$$f(x + f(y)) = f(x) + y$$

সত্য হয়।

Solution:

First, we prove that f is bijective. Setting $x = 0$ gives $f(f(y)) = f(0)+y$. If $f(a) = f(b)$, then $f(0) + a = f(f(a)) = f(f(b)) = f(0) + b$, which implies $a = b$. Thus, f is injective. Moreover, taking $y = z - f(0)$ gives $f(f(z - f(0))) = z$, proving f is surjective.

Now, set $y = 0$ in the original equation:

$$f(x + f(0)) = f(x) + 0$$

Because f is injective, $f(x + f(0)) = f(x)$ directly implies $x + f(0) = x$, meaning $f(0) = 0$. Now our earlier bijection relation $f(f(y)) = f(0) + y$ simplifies to:

$$f(f(y)) = y$$

Substitute y with $f(y)$ in the original equation. Since $f(f(y)) = y$, we get:

$$f(x + f(f(y))) = f(x) + f(y) \implies f(x + y) = f(x) + f(y)$$

This proves f satisfies Cauchy's functional equation. Since we are given that f is continuous, the only solutions to Cauchy's equation are linear functions of the form $f(x) = cx$ for some real constant c.

Substitute $f(x) = cx$ into the original equation:

$$c(x + cy) = cx + y \implies cx + c^2y = cx + y$$

Matching the coefficients of y, we get $c^2 = 1$, which gives $c = 1$ or $c = -1$.

Thus, the only solutions are $\boxed{f(x) = x}$ and $\boxed{f(x) = -x}$. Both can be easily verified to satisfy the original equation. ■

Problem 2 (Number Theory)**.** *Find all positive integers $n > 1$ such that n^2 divides $2^n + 1$.*
[সংখ্যাতত্ত্ব] এমন সকল ধনাত্মক পূর্ণসংখ্যা $n > 1$ নির্ণয় কর যার জন্য $2^n + 1$ সংখ্যাটি n^2 দ্বারা বিভাজ্য।

Solution:

We claim $n = 3$ is the only solution.

If $n > 1$, then since $2^n + 1$ is always odd, n must be an odd integer. Let p be the smallest prime factor of n. Since n is odd, $p \geq 3$.

Since $p \mid n$ and $n^2 \mid 2^n + 1$, we have $p \mid 2^n + 1$. This implies $2^n \equiv -1 \pmod p$, and squaring yields $2^{2n} \equiv 1 \pmod p$. By Fermat's Little Theorem, we also know $2^{p-1} \equiv 1 \pmod p$.

Let d be the multiplicative order of 2 modulo p. We have $d \mid 2n$ and $d \mid p - 1$. Thus, d must divide $\gcd(2n, p - 1)$. Because p is the smallest prime dividing n, n shares no prime factors with $p - 1$. Hence $\gcd(n, p - 1) = 1$. This implies $\gcd(2n, p - 1)$ divides 2, so $d \mid 2$.

Thus, either $d = 1$ or $d = 2$. This means $2^1 \equiv 1 \pmod p$ or $2^2 \equiv 1 \pmod p$, which forces $p \mid 1$ or $p \mid 3$. The only prime satisfying this is $p = 3$.

Since $p = 3$ is the smallest prime factor, we can write $n = 3^k q$ where $\gcd(q, 3) = 1$. By the Lifting the Exponent Lemma (since $3 \mid 2 + 1$):

$$v_3(2^n + 1) = v_3(2 + 1) + v_3(n) = 1 + k$$

Because $n^2 \mid 2^n + 1$, we must have $v_3(n^2) \leq v_3(2^n + 1)$, which means $2k \leq 1 + k \implies k \leq 1$. Since $p = 3 \mid n$, we have $k \geq 1$. Thus $k = 1$. This gives $n = 3q$ with $\gcd(q, 3) = 1$.

If $q > 1$, let r be the smallest prime factor of q. Since $p = 3$ was the smallest prime factor of n, $r > 3$. We have $r \mid 2^{3q} + 1 = 8^q + 1$. By the same logic, $8^{2q} \equiv 1 \pmod r$ and $8^{r-1} \equiv 1 \pmod r$. The order of 8 modulo r divides $\gcd(2q, r - 1) = 2$ (since r is the smallest prime factor of q, $\gcd(q, r - 1) = 1$). So $8^2 \equiv 1 \pmod r$, meaning $r \mid 63$. The prime factors of 63 are 3 and 7. Since $r > 3$, we must have $r = 7$. But $7 \mid 2^{3q} + 1 = 8^q + 1 \equiv 1^q + 1 = 2 \pmod 7$, absolute contradiction.

Thus $q = 1$, leaving $\boxed{n = 3}$ as the unique answer. ■

Problem 3 (Combinatorics)**.** *Let S be a set of n points in the plane ($n \geq 2$) such that the distance between any two distinct points is at least 1. Prove that the number of pairs (A, B) of points in S whose distance is exactly 1 is at most $3n$.*

[সমবায়বিদ্যা] ধরি S হলো সমতলে n টি বিন্দুর একটি সেট ($n \geq 2$) যেখানে যেকোনো দুটি ভিন্ন বিন্দুর মধ্যবর্তী দূরত্ব অন্তত 1। প্রমাণ কর যে S সেটের এমন বিন্দুজোড় (A, B) এর সর্বোচ্চ সংখ্যা $3n$ হতে পারে, যাদের মধ্যবর্তী দূরত্ব ঠিক 1।

Solution:

We construct a graph $G = (V, E)$ where the vertices are the n points in S. We connect two vertices with an edge if and only if the distance between them is exactly 1. The number of edges $|E|$ is the number of such pairs.

Consider any arbitrary point $P \in S$. Let $N(P)$ be the set of points in S connected to P. By our definition, every point in $N(P)$ is exactly distance 1 away from P. Thus, all points in $N(P)$ lie on a circle of radius 1 centered at P.

We are given that the distance between any two points in S is at least 1. Therefore, for any two distinct points $A, B \in N(P)$, the distance between A and B must be ≥ 1. In a circle of radius 1, a chord of length 1 corresponds to an arc of 60°. If two points on the circle are separated by a distance of at least 1, the angle subtended by them at the center P must be at least 60°.

Because the total angle around the center P is 360°, there can be at most $360°/60° = 6$ points arranged on this circle such that any two subtend at least 60° at the center. This means $|N(P)| \leq 6$.

In graph-theoretic terms, the degree of every vertex in G is at most 6. By the Handshake Lemma, the sum of the degrees of all vertices is equal to twice the number of edges:

$$2|E| = \sum_{v \in V} \deg(v)$$

Since $\deg(v) \leq 6$ for each of the n vertices, we have:

$$2|E| \leq 6n \implies |E| \leq 3n$$

Thus, there are at most $3n$ pairs of points at distance exactly 1. ■

Problem 4 (Geometry)**.** *Let ABC be a triangle with circumcenter O and incenter I. Let R be the circumradius and r be the inradius of $\triangle ABC$. Prove Euler's Formula:*

$$OI^2 = R^2 - 2Rr$$

[জ্যামিতি] ধরি ABC একটি ত্রিভুজ যার পরিকেন্দ্র O এবং অন্তঃকেন্দ্র I। ত্রিভুজটির পরিব্যাসার্ধ R এবং অন্তর্ব্যাসার্ধ r হলে অয়লারের সূত্রটি (Euler's Formula) প্রমাণ কর:

$$OI^2 = R^2 - 2Rr$$

Solution:

Let the circumcircle of $\triangle ABC$ be Γ. Extend the angle bisector AI to intersect Γ again at L. By the Power of a Point Theorem applied to point I concerning the circumcircle Γ:

$$R^2 - OI^2 = IA \cdot IL$$

We now establish the lengths of IA and IL. Let F be the projection of the incenter I onto AB. In the right-angled $\triangle AIF$, we have $\angle IAF = A/2$. By trigonometry:

$$IA = \frac{IF}{\sin(A/2)} = \frac{r}{\sin(A/2)}$$

Next, we look at the point L. By the Incenter-Excenter Lemma (also known as the "Fact 5" or "Trident" lemma), the point L, being the intersection of the angle bisector of A and the circumcircle, is the center of the circle passing through B, I, C, and the excenter I_a. Therefore, $\triangle IBL$ is isosceles and $IL = BL$.

To find the length of BL, we apply the Extended Law of Sines in $\triangle ABL$ inscribed in the circumcircle Γ of radius R. The angle subtended by chord BL at the circumference is $\angle BAL = A/2$. Thus:

$$BL = 2R\sin(A/2)$$

Since $IL = BL$, we have $IL = 2R\sin(A/2)$.

Substitute the established lengths into the Power of a Point equation:

$$\begin{aligned} R^2 - OI^2 &= IA \cdot IL \\ &= \left(\frac{r}{\sin(A/2)}\right) \cdot (2R\sin(A/2)) \\ &= 2Rr \end{aligned}$$

Rearranging the terms, we get the desired formula:

$$OI^2 = R^2 - 2Rr$$

■