Problem Set

Problem 1: A Divisibility Condition

Problem 1. *Find all positive integers n such that $n+3$ divides n^2+1.*
এমন সকল ধনাত্মক পূর্ণসংখ্যা n নির্ণয় কর, যাতে $n+3$ সংখ্যাটি n^2+1-কে ভাগ করে।

Answer

Answer: $n = 2$ and $n = 7$.

Solution: We have

$$n^2 + 1 = (n+3)(n-3) + 10.$$

Therefore $n+3 \mid n^2+1$ if and only if $n+3 \mid 10$. Since n is positive, $n+3 \geq 4$, so the only possible divisors of 10 in this range are 5 and 10. Hence $n+3=5$ or $n+3=10$, giving $n=2$ or $n=7$. Both work: $5 \mid 5$ and $10 \mid 50$. ■

Problem 2: A Recurrence to Sum

Problem 2. *Let* $a_1 = 1$ *and* $a_{n+1} = 3a_n + 2$ *for all* $n \geq 1$. *Find the sum* $a_1 + a_2 + \cdots + a_{2026}$.
ধরি $a_1 = 1$, এবং সকল $n \geq 1$-এর জন্য $a_{n+1} = 3a_n + 2$। এখন $a_1 + a_2 + \cdots + a_{2026}$ যোগফল নির্ণয় কর।

Answer

Answer: $3^{2026} - 2027$.

Solution: Adding 1 to both sides of the recurrence,

$$a_{n+1} + 1 = 3(a_n + 1).$$

Hence the sequence $a_n + 1$ is geometric with first term $a_1 + 1 = 2$ and ratio 3:

$$a_n + 1 = 2 \cdot 3^{n-1}, \qquad \text{so} \qquad a_n = 2 \cdot 3^{n-1} - 1.$$

Summing from $n = 1$ to 2026,

$$\sum_{n=1}^{2026} a_n = 2 \sum_{n=1}^{2026} 3^{n-1} - 2026 = 2 \cdot \frac{3^{2026} - 1}{3 - 1} - 2026 = 3^{2026} - 1 - 2026 = 3^{2026} - 2027. \quad \blacksquare$$

Problem 3: Avoiding Distance Two

Problem 3. *Find the number of nonempty subsets of* $\{1, 2, \ldots, 20\}$ *that contain no two integers differing by* 2.

$\{1, 2, \ldots, 20\}$-এর এমন অশূন্য উপসেটের সংখ্যা নির্ণয় কর, যাতে এমন দুটি পূর্ণসংখ্যা নেই যাদের পার্থক্য 2।

Answer

Answer: 20735.

Solution: Let F_m denote the number of subsets of a path on m vertices that contain no two adjacent vertices, counting the empty set. A path on m vertices has $F_m = F_{m-1} + F_{m-2}$ with $F_1 = 2$, $F_2 = 3$, so F_m is a Fibonacci-type number; explicitly F_m counts independent sets.

Now connect two numbers in $\{1, \ldots, 20\}$ if they differ by 2. This graph splits into two disjoint paths: the odd numbers $1, 3, 5, \ldots, 19$ and the even numbers $2, 4, 6, \ldots, 20$, each with 10 vertices. A subset is allowed precisely when it is independent in both paths, so the number of allowed subsets, counting the empty set, is

$$F_{10} \cdot F_{10}.$$

Since $F_1 = 2, F_2 = 3, F_3 = 5, \ldots$, we have $F_{10} = 144$. Hence the total number of allowed subsets, including the empty set, is $144^2 = 20736$, and the number of nonempty ones is

$$20736 - 1 = \boxed{20735}. \quad \blacksquare$$

Problem 4: Twin Primes by Twos

Problem 4. *Find all primes p such that both p^2+2 and p^2+4 are prime.*
এমন সকল মৌলিক সংখ্যা p নির্ণয় কর, যাতে p^2+2 এবং p^2+4 উভয়ই মৌলিক হয়।

Answer

Answer: $p = 3$.

Solution: For $p = 3$ we get $p^2 + 2 = 11$ and $p^2 + 4 = 13$, both prime. We show this is the only possibility.
If $p = 2$, then $p^2 + 2 = 6$ and $p^2 + 4 = 8$ are not prime. Suppose now p is an odd prime. Modulo 3, the only nonzero square is 1, so $p^2 \equiv 1 \pmod 3$ and therefore

$$p^2 + 2 \equiv 0 \pmod 3.$$

For $p \geq 5$ the number $p^2 + 2 > 3$ is divisible by 3 and is composite. Hence no $p \geq 5$ works, and the only solution is $\boxed{p = 3}$. ■

Problem 5: A Recurrence for Functions

Problem 5. *Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that*

$$f(n) + f(n+1) = 2n + 3$$

for every positive integer n.
এমন সকল ফাংশন $f : \mathbb{N} \to \mathbb{N}$ নির্ণয় কর, যাতে প্রতিটি ধনাত্মক পূর্ণসংখ্যা n-এর জন্য $f(n)+f(n+1) = 2n+3$ হয়।

Answer

Answer: Exactly the functions

$$f(n) = n + c \text{ for odd } n, \qquad f(n) = n + 2 - c \text{ for even } n,$$

where $c \in \{0, 1, 2, 3\}$.

Solution: Writing the given identity for n and for $n+1$ and subtracting,

$$f(n+2) - f(n) = 2.$$

Hence $f(n) - n$ depends only on the parity of n. So there are integers c_o, c_e with

$$f(n) = n + c_o \ (n \text{ odd}), \qquad f(n) = n + c_e \ (n \text{ even}).$$

Substituting into $f(n) + f(n+1) = 2n + 3$ (either for n odd or n even) gives $c_o + c_e = 2$. Since f takes positive values,

$$f(1) = 1 + c_o \geq 1 \ \Rightarrow \ c_o \geq 0, \qquad f(2) = 2 + c_e \geq 1 \ \Rightarrow \ c_e \geq -1.$$

With $c_o + c_e = 2$ this yields $c_o \in \{0, 1, 2, 3\}$, and $c_e = 2 - c_o$. Setting $c = c_o$ gives exactly the four displayed functions, each of which is readily checked to satisfy the equation. ■

Problem 6: Points on a Circle Modulo p

Problem 6. *Let p be an odd prime. Determine the number of ordered pairs (x, y) with $x, y \in \{0, 1, \ldots, p-1\}$ such that*

$$x^2 + y^2 \equiv 1 \pmod{p}.$$

ধরি p একটি বিজোড় মৌলিক সংখ্যা। $x, y \in \{0, 1, \ldots, p-1\}$ এমন ক্রমজোড় (x, y)-এর সংখ্যা নির্ণয় কর, যাতে $x^2 + y^2 \equiv 1 \pmod{p}$ হয়।

Answer

Answer: $p - 1$ if $p \equiv 1 \pmod{4}$, and $p + 1$ if $p \equiv 3 \pmod{4}$.

Solution: Let χ denote the Legendre symbol modulo p, with $\chi(0) = 0$. For a fixed x, the congruence $y^2 \equiv 1 - x^2 \pmod{p}$ has exactly $1 + \chi(1 - x^2)$ solutions in y. Hence the total number N of pairs is

$$N = \sum_{x \in \mathbb{F}_p} \bigl(1 + \chi(1 - x^2)\bigr) = p + \sum_x \chi(1 - x^2).$$

Now $\chi(1 - x^2) = \chi(-1)\,\chi(x^2 - 1)$, so

$$N = p + \chi(-1) \sum_x \chi(x^2 - 1).$$

We compute $\sum_x \chi(x^2 - 1)$. The equation $y^2 = x^2 - 1$ is equivalent to $(x - y)(x + y) = -1$. For each of the $p - 1$ values of $x - y \in \mathbb{F}_p^{\times}$, the value of $x + y$ is determined uniquely, so this equation has exactly $p - 1$ solutions. On the other hand, for each x it has $1 + \chi(x^2 - 1)$ solutions in y. Thus

$$p - 1 = \sum_x \bigl(1 + \chi(x^2 - 1)\bigr) = p + \sum_x \chi(x^2 - 1),$$

so $\sum_x \chi(x^2 - 1) = -1$. Therefore

$$N = p - \chi(-1) = p - (-1)^{(p-1)/2},$$

i.e. $N = p - 1$ for $p \equiv 1 \pmod{4}$ and $N = p + 1$ for $p \equiv 3 \pmod{4}$. ■

Problem 7: Monochromatic Triangles

Problem 7. *Every edge of the complete graph K_6 is colored either red or blue. Prove that there are at least two monochromatic triangles.*
সম্পূর্ণ গ্রাফ K_6-এর প্রতিটি ধার হয় লাল নয়তো নীল রঙ করা হয়েছে। প্রমাণ কর যে, অন্তত দুটি একরঙা ত্রিভুজ বিদ্যমান।

Solution

Let r_i and b_i denote the numbers of red and blue edges meeting at vertex i, so that $r_i + b_i = 5$. Count the *monochromatic angles*, i.e. pairs of edges of the same color sharing a vertex. At vertex i the number of such pairs is

$$\binom{r_i}{2} + \binom{b_i}{2}.$$

On the other hand, a monochromatic triangle contains three monochromatic angles, while a triangle that is not monochromatic (which must have exactly two edges of one color and one of the other) contains exactly one. Since K_6 has $\binom{6}{3} = 20$ triangles, if t denotes the number of monochromatic triangles, then

$$3t + (20 - t) = \sum_{i=1}^{6} \left(\binom{r_i}{2} + \binom{b_i}{2} \right).$$

For each vertex, since $r_i + b_i = 5$, the quantity $\binom{r_i}{2} + \binom{b_i}{2}$ is at least 4 (attained at $\{r_i, b_i\} = \{2, 3\}$). Hence the right side is at least $6 \cdot 4 = 24$, and

$$2t + 20 \geq 24 \quad \Longrightarrow \quad t \geq 2.$$

So every such coloring contains at least two monochromatic triangles. ■

Problem 8: A Weighted Sum of Fractions

Problem 8. *Let a, b, c be positive real numbers with $abc = 1$. Prove that*

$$\frac{a^2+b^2}{a+b}+\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a}\geq 3.$$

ধরি a, b, c এমন ধনাত্মক বাস্তব সংখ্যা যাতে $abc = 1$। প্রমাণ কর যে,

$$\frac{a^2+b^2}{a+b}+\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a}\geq 3.$$

Solution

For any positive x, y, the inequality $2(x^2+y^2) \geq (x+y)^2$ gives

$$\frac{x^2+y^2}{x+y}\geq\frac{x+y}{2}.$$

Applying this to each summand,

$$\frac{a^2+b^2}{a+b}+\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a}\geq\frac{(a+b)+(b+c)+(c+a)}{2}=a+b+c.$$

Finally, by AM–GM, $a+b+c \geq 3\sqrt[3]{abc} = 3$. Combining the two inequalities gives the result. Equality holds at $a = b = c = 1$. ■

Problem 9: Parallel Feet

Problem 9. *Let ABC be an acute triangle. Let D and E be the feet of the altitudes from B and C, respectively. Prove that the line DE is parallel to BC if and only if $AB = AC$.*
ধরি ABC একটি সূক্ষ্মকোণী ত্রিভুজ। D ও E যথাক্রমে B ও C থেকে টানা লম্বের পাদবিন্দু। প্রমাণ কর যে, DE রেখা BC-এর সমান্তরাল হবে যদি এবং কেবল যদি $AB = AC$ হয়।

Solution

Let $\beta = \angle ABC$ and $\gamma = \angle ACB$. Since $\angle BDC = \angle BEC = 90°$, the four points B, C, D, E are concyclic (on the circle with diameter BC).
By the inscribed-angle theorem applied to this circle,

$$\angle BDE = \angle BCE.$$

Because $CE \perp AB$, the angle between CE and CB is $90° - \beta$, so $\angle BCE = 90° - \beta$. Hence

$$\angle BDE = 90° - \beta. \tag{1}$$

On the other hand, the angle between DB and BC is $90° - \gamma$, since $DB \perp AC$. Using directed angles modulo 180°, we have

$$DE \parallel BC \iff \angle BDE = \angle(DB, BC) = 90° - \gamma.$$

Combined with (1), this holds exactly when

$$90° - \beta = 90° - \gamma \iff \beta = \gamma \iff AB = AC,$$

the last equivalence holding because equal angles are opposite equal sides. ■

Problem 10: Subsets with Sum Divisible by p

Problem 10. *Let p be an odd prime. Find the number of nonempty subsets of $\{1, 2, \ldots, p-1\}$ whose elements add up to a multiple of p.*

ধরি p একটি বিজোড় মৌলিক সংখ্যা। $\{1, 2, \ldots, p-1\}$-এর এমন অশূন্য উপসেটের সংখ্যা নির্ণয় কর, যাদের উপাদানগুলোর যোগফল p-এর গুণিতক।

Answer

Answer: $\dfrac{2^{p-1}-1}{p}$.

Solution: Let ζ be a primitive p-th root of unity. For an integer m, the sum $\sum_{j=0}^{p-1} \zeta^{jm}$ equals p if $p \mid m$ and 0 otherwise. Hence, if N is the number of subsets of $\{1, \ldots, p-1\}$ (including the empty set) whose sum is divisible by p, then

$$N = \frac{1}{p} \sum_{j=0}^{p-1} \sum_{S \subseteq \{1,\ldots,p-1\}} \zeta^{j \sum S} = \frac{1}{p} \sum_{j=0}^{p-1} \prod_{i=1}^{p-1} \left(1 + \zeta^{ji}\right).$$

For $j = 0$ the product is 2^{p-1}. For $j \neq 0$, as i runs over $1, \ldots, p-1$ so does ji modulo p, and therefore

$$\prod_{i=1}^{p-1} \left(1 + \zeta^{ji}\right) = \prod_{i=1}^{p-1} \left(1 + \zeta^{i}\right).$$

Since $1 + x + \cdots + x^{p-1} = \prod_{i=1}^{p-1} (x - \zeta^i)$, putting $x = -1$ gives

$$\prod_{i=1}^{p-1} (-1 - \zeta^i) = \frac{(-1)^p - 1}{-2} = 1,$$

and as $p-1$ is even, this product equals $\prod_{i=1}^{p-1} (1 + \zeta^i)$. Thus each of the $p-1$ products with $j \neq 0$ equals 1. Consequently

$$N = \frac{2^{p-1} + (p-1)}{p}.$$

Subtracting the empty set, the required number of nonempty subsets is

$$N - 1 = \boxed{\frac{2^{p-1}-1}{p}}. \quad \blacksquare$$