Friends, Groups, and Societies Problem

Problem 1. *Sehtaj has 10001 friends. Some of them join together to form several* groups *(a friend may belong to different groups). Some groups join together to form several* societies *(a group may belong to different societies). There are a total of k societies. Suppose that the following conditions hold:*

(i) Each pair of friends is in exactly one group.

(ii) For each friend and each society, the friend is in exactly one group of the society.

(iii) Each group has an odd number of friends. In addition, a group with $2m+1$ friends (m is a positive integer) is in exactly m societies.

Find all possible values of k.

Answer

In order to use all the information in the question, we count triples (a, C, S), where a is a friend, C is a group and S is a society, where $a \in C$ and $C \in S$. Let the number of such triples be T.

Suppose we first fix a, then S, then C. We can choose a in 10001 ways, S in k ways and then finally C in only one way (by condition (ii)). Hence $T = 10001 \cdot k$.

Now suppose we fix C first. There are $|C|$ ways of doing this. Then by condition (iii), there are $(|C|-1)/2$ ways to choose S. Finally there is only one way to choose a, by (ii). This gives

$$T = \sum_{C \text{ is a group}} |C|(|C|-1)/2 = \sum_{C \text{ is a group}} \binom{|C|}{2}$$

On the other hand, the sum $\displaystyle\sum_{C \text{ is a group}} \binom{|C|}{2}$ is actually equal to the number of pairs of friends. This is because each pair of friends is in exactly one group by (i), so each pair of friends is counted exactly once. Hence this sum is equal to $\binom{10001}{2}$, so putting everything together

$$\binom{10001}{2} = T = 10001 \cdot k \implies k = 5000.$$

Finally, to construct a configuration for $k = 5000$, let there be only one group (C) containing all 10001 friends, and let this group be in all 5000 societies. Since $|C| = 10001 =$

2(5000)+1, by condition (iii), this group should be in exactly 5000 societies, which is satisfied. Condition (i) is satisfied since all pairs of friends are in the one group C. Condition (ii) is satisfied since for each friend and each society, the friend is in exactly one group (namely C) of that society. ■